

# SPECTRAL GAP AND QUANTITATIVE STATISTICAL STABILITY FOR SYSTEMS WITH CONTRACTING FIBERS AND LORENZ-LIKE MAPS.

STEFANO GALATOLO AND RAFAEL LUCENA

**ABSTRACT.** We consider transformations preserving a contracting foliation, such that the associated quotient map satisfies a Lasota Yorke inequality. We prove that the associated transfer operator, acting on suitable normed spaces, has a spectral gap (on which we have quantitative estimation).

As an application we consider Lorenz-Like two dimensional maps (piecewise hyperbolic with unbounded contraction and expansion rate): we prove that those systems have a spectral gap and we show a quantitative estimation for their statistical stability. Under deterministic perturbations of the system of size  $\delta$ , the physical measure varies continuously, with a modulus of continuity  $O(\delta \log \delta)$ .

## 1. INTRODUCTION

The study of the behavior of the transfer operator restricted to a suitable functional space has proven to be a powerful tool for the understanding of the statistical properties of a dynamical system. This approach gave first results in the study of the dynamics of piecewise expanding maps where the involved spaces are made of regular, absolutely continuous measures (see [5], [22], [28] for some introductory text). In recent years the approach was extended to piecewise hyperbolic systems by the use of suitable anisotropic norms (the expanding and contracting directions are managed differently), leading to suitable distribution spaces on which the transfer operator has good spectral properties (see e.g. [7], [6], [10], [17]). From these properties, several limit theorems or stability statements can be deduced. This approach has proven to be successful in non-trivial classes of systems like geodesic flows (see [22],[9]) or billiard maps (ess e.g. [12] [13] where a relatively simple and unified approach to many limit and perturbative results is given for the Lorentz gas). We remark that in these approaches, usually some condition of boundedness of the derivatives or transversality between the map's singular set and the contracting directions is supposed.

In this work, we consider skew product maps preserving a uniformly contracting foliation. We show how it is possible, in a simple way, to define suitable spaces of signed measures (with an anisotropic norm) such that, under small regularity assumptions, the transfer operator associated to the dynamics has a spectral gap (in the sense given in Theorem 6.1). This shows an exponential convergence to 0 in a certain norm for the iteration of a large class of zero average measures by the transfer operator. We remark that in this approach the speed of this convergence

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*Date:* December 23, 2016.

*Key words and phrases.* Spectral Gap, Statistical Properties, Lorenz-Like, Transfer Operator, Stability.

can be quantitatively estimated, and depends on the rate of contraction of the stable foliation, the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of the induced quotient map (see Remark 6.3). We also remark that in our approach we can deal with maps having  $C^{1+\alpha}$  regularity, having unbounded derivatives, and where the singular set is parallel to the contracting direction, as it happens in the Lorenz-like maps we consider in Section 7.

The function spaces we consider are defined by disintegrating signed measures on the phase space along the contracting foliation. The signed measure itself is then seen as a family of measures on the contracting leaves. We can then consider some notion of regularity for this family to define suitable spaces of more or less “regular” measures where to apply our transfer operator. To give an idea of these function spaces (see section 3), in the case of skew product maps of the unit square  $I \times I$  to itself, the disintegration gives rise to a one dimensional family (a path) of measures defined on the contracting leaves, each leaf is isomorphic to the unit interval  $I$ , hence a measure on  $I \times I$  is seen as a path of measures on  $I$ : a path in a metric space. The function spaces are defined by suitable notions of regularity for these paths. In the case  $I \times I$  for example, the spaces which arise are included in  $L^1(I, Lip(I)')$  (the space of  $L^1$  functions from the interval to the dual of the space of Lipschitz functions on the interval), imposing some kind of further regularity. We remark that this is a space of distribution valued functions. For simplicity we will only use normed vector spaces of signed measures in this paper, we do not need to consider the completion of the space of signed measure, which would lead to distribution spaces.

The paper is structured as follows: in Section 3 we introduce the functional spaces we consider; in Section 4 we show the basic properties of the transfer operator when applied to these spaces. In particular we see that there is a useful “Perron-Frobenius”-like formula. In Section 5 we see the basic properties of the iteration of the transfer operator on the spaces we consider. In particular we see Lasota-Yorke inequalities and a convergence to equilibrium statement. In Section 6 we use the convergence to equilibrium and the Lasota-Yorke inequalities to prove the spectral gap. In Section 7 we present an application of our construction, showing a spectral gap for 2-dimensional Lorenz-like maps (piecewise  $C^{1+\alpha}$  hyperbolic maps with unbounded expansion and contraction rates). In Section 8 we apply our construction to a class of piecewise  $C^2$  Lorenz-like maps. We prove stronger (bounded variation) regularity results for the iteration of probability measures on that systems, and use this to prove a strong statistical stability statement with respect to deterministic perturbations: we establish a modulus of continuity  $\delta \log \delta$  for the variation of the physical measure in weak space  $(L^1(I, Lip(I)'))$  after a “size  $\delta$ ” perturbation. We remark that a qualitative statement, for a class of similar maps was given in [1].

**Acknowledgment** This work was partially supported by FAPEAL (Brazil) Grants 60030 000587/2016, CNPq (Brazil) Grants 300398/2016-6, CAPES (Brazil) Grants 99999.014021/2013-07 and EU Marie-Curie IRSES Brazilian-European partnership in Dynamical Systems (FP7-PEOPLE- 2012-IRSES 318999 BREUDS).

## 2. CONTRACTING FIBER MAPS

Consider  $\Sigma = N_1 \times N_2$ , where  $N_1$  and  $N_2$  are compact and finite dimensional Riemannian manifolds such that  $\text{diam}(N_2) = 1$ , where  $\text{diam}(N_2)$  denotes the diameter of  $N_2$  with respect to its Riemannian metric,  $d_2$ . This is not restrictive but will avoid some multiplicative constants. Denote by  $m_1$  and  $m_2$  the Lebesgue measures on  $N_1$  and  $N_2$  respectively, generated by their corresponding Riemannian volumes, normalized so that  $m_1(N_1) = m_2(N_2) = 1$  and  $m = m_1 \times m_2$ . Consider a map  $F : (\Sigma, m) \longrightarrow (\Sigma, m)$ ,

$$(1) \quad F(x, y) = (T(x), G(x, y)),$$

where  $T : N_1 \longrightarrow N_1$  and  $G : \Sigma \longrightarrow N_2$  are measurable maps. Suppose that these maps satisfy the following conditions

2.0.1. *Properties of  $G$ .*

**G1:** Consider the  $F$ -invariant foliation

$$(2) \quad \mathcal{F}^s := \{\{x\} \times N_2\}_{x \in N_1}.$$

We suppose that  $\mathcal{F}^s$  is contracted: there exists  $0 < \alpha < 1$  such that for all  $x \in N_1$  it holds

$$(3) \quad d_2(G(x, y_1), G(x, y_2)) \leq \alpha d_2(y_1, y_2), \quad \text{for all } y_1, y_2 \in N_2.$$

2.0.2. *Properties of  $T$  and of its associated transfer operator.* Suppose that:

**T1:**  $T$  is non-singular with respect to  $m_1$  ( $m_1(A) = 0 \Rightarrow m_1(T^{-1}(A)) = 0$ ).

**T2:** There exists a disjoint collection of open sets  $\mathcal{P} = \{P_1, \dots, P_q\}$  of  $N_1$ , such that  $m_1(\bigcup_{i=1}^q P_i) = 1$  and  $T_i := T|_{P_i}$  is a diffeomorphism :  $P_i \rightarrow T_i(P_i) \subseteq N_1$ , with  $\det DT_i(x) \neq 0$  for all  $x \in P_i$  and for all  $i$ , where  $DT_i$  is the Jacobian of  $T_i$  with respect to the Riemannian metric of  $N_1$ .

**T3:** Let us consider the Perron-Frobenius Operator associated to  $T$ ,  $P_T^{-1}$ . We will now make some assumption on the existence of a suitable functional analytic setting adapted to  $P_T$ . Let us hence denote the  $L_{m_1}^1$  norm<sup>2</sup> by  $|\cdot|_1$  and suppose that there exists a Banach space  $(S_-, |\cdot|_s)$  such that

**T3.1:**  $S_- \subset L_{m_1}^1$  is  $P_T$ -invariant,  $|\cdot|_1 \leq |\cdot|_s$  and  $P_T : S_- \longrightarrow S_-$  is bounded;

**T3.2:** The unit ball of  $(S_-, |\cdot|_s)$  is relatively compact in  $(L_{m_1}^1, |\cdot|_1)$ ;

**T3.3:** (Lasota Yorke inequality) There exists  $k \in \mathbb{N}$ ,  $0 < \beta_0 < 1$  and  $C > 0$  such that, for all  $f \in S_-$ , it holds

$$(4) \quad |P_T^k f|_s \leq \beta_0 |f|_s + C |f|_1.$$

**T3.4:** Suppose there is an unique  $\psi_x \in S_-$  with  $\psi_x \geq 0$  and  $|\psi_x|_1 = 1$  such that  $P_T(\psi_x) = \psi_x$ , and if  $\psi \in S_-$  is another density for a probability measure, then  $P_T^k(\psi_x - \psi) \rightarrow 0$  in  $S_-$ .<sup>3</sup>

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<sup>1</sup>The unique operator  $P_T : L_{m_1}^1 \longrightarrow L_{m_1}^1$  such that

$$\forall \phi \in L_{m_1}^1 \quad \text{and} \quad \forall \psi \in L_{m_1}^\infty \quad \int \psi \cdot P_T(\phi) \, dm_1 = \int (\psi \circ T) \cdot \phi \, dm_1.$$

<sup>2</sup>**Notation:** In the following we use  $|\cdot|$  to indicate the usual absolute value or norms for signed measures on the basis space  $N_1$ . We will use  $\|\cdot\|$  for norms defined for signed measures on  $\Sigma$ .

<sup>3</sup>This assumption ensures that from our point of view the system is indecomposable. For piecewise expanding maps e.g., the assumption follows from topological mixing.

It is known that in this case ([20], see also [28], [22] ) the following holds.

**2.1. Theorem.** *If  $T$  satisfy T3.1, ..., T3.4 then there exist  $0 < r < 1$  and  $D > 0$  such that for all  $\phi \in S_-$  with  $\int \phi \, dm_1 = 0$  and for all  $n \geq 0$ , it holds*

$$(5) \quad |\mathbf{P}_T^n(\phi)|_s \leq Dr^n |\phi|_s.$$

The following additional property on  $|\cdot|_s$  will be supposed sometimes in the paper, to obtain spectral gap on  $L^\infty$  like spaces.

**N1:** There is  $H_N \geq 0$  such that  $|\cdot|_\infty \leq H_N |\cdot|_s$  (where  $|\cdot|_\infty$  is the usual  $L_{m_1}^\infty$  norm on  $N_1$  )

Iterating the inequality (4) and since it holds  $|\mathbf{P}_T(h)|_1 \leq |h|_1$ , for all  $h \in L_{m_1}^1$ , we have

$$(6) \quad |\mathbf{P}_T^{lk} f|_s \leq \beta_0^l |f|_s + \frac{C}{1 - \beta_0} |f|_1,$$

for all  $f \in S_-$  and for all  $l \in \mathbb{N}$ . For a given  $n \in \mathbb{N}$ , set  $n = q_n k + r_n$ , where  $0 \leq r_n \leq k$ . Since  $\mathbf{P}_T : S_- \rightarrow S_-$  is bounded, there exists  $M_1 > 0$  such that  $|\mathbf{P}_T^{r_n} f|_s \leq M_1$  for all  $n$ , where  $|\mathbf{P}_T^{r_n} f|_s = \sup_{f \in S_-, f \neq 0} \frac{|\mathbf{P}_T^{r_n}(f)|_s}{|f|_s}$ . Thus, we have

$$\begin{aligned} |\mathbf{P}_T^n f|_s &= |\mathbf{P}_T^{q_n k + r_n} f|_s \\ &= |\mathbf{P}_T^{q_n k}(\mathbf{P}_T^{r_n} f)|_s \\ &\leq \beta_0^{q_n} |\mathbf{P}_T^{r_n} f|_s + \frac{C}{1 - \beta_0} |f|_1 \\ &\leq \beta_0^{q_n} M_1 |f|_s + \frac{C}{1 - \beta_0} |f|_1 \\ &\leq \beta_0^{\frac{n - r_n}{k}} M_1 |f|_s + \frac{C}{1 - \beta_0} |f|_1 \\ &\leq \left(\beta_0^{\frac{1}{k}}\right)^n \frac{M_1}{\beta_0} |f|_s + \frac{C}{1 - \beta_0} |f|_1 \end{aligned}$$

Setting  $B_3 = \frac{M_1}{\beta_0}$ ,  $\beta_2 = \beta_0^{\frac{1}{k}}$  and  $C_2 = \frac{C}{1 - \beta_0}$ , we get

$$(7) \quad |\mathbf{P}_T^n f|_s \leq B_3 \beta_2^n |f|_s + C_2 |f|_1, \quad \forall n, \quad \forall f \in S_-,$$

where  $0 < \beta_2 < 1$ .

### 3. WEAK AND STRONG SPACES

**3.1.  $L^1$  like spaces.** Through this section we construct some function spaces which are suitable for the systems we consider. The idea is to consider spaces of signed measures, with suitable norms constructed by disintegrating measures along the stable foliation. Thus a signed measure will be seen as a family of measures on each leaf. As an example, a measure on the square will be seen as a one parameter family (a path) of measures on the interval (a stable leaf). In the vertical, contracting direction (on the leaves) we will consider a norm which is the dual of the Lipschitz norm. In the ‘‘horizontal’’ direction we will consider essentially the  $L^1$  norm.

Consider a probability space  $(\Sigma, \mathcal{B}, \mu)$  and a partition  $\Gamma$  of  $\Sigma$  by measurable sets  $\gamma \in \mathcal{B}$ . Denote by  $\pi : \Sigma \rightarrow \Gamma$  the projection that associates to each point  $x \in M$

the element of  $\Gamma$  which contains  $x$ , i.e.  $\pi(x) = \gamma_x$ . Let  $\widehat{\mathcal{B}}$  be the  $\sigma$ -algebra of  $\Gamma$  provided by  $\pi$ . Precisely, a subset  $\mathcal{Q} \subset \Gamma$  is measurable if, and only if,  $\pi^{-1}(\mathcal{Q}) \in \mathcal{B}$ . We define the *quotient* measure  $\mu_x$  on  $\Gamma$  by  $\mu_x(\mathcal{Q}) = \mu(\pi^{-1}(\mathcal{Q}))$ .

The proof of the following theorem can be found in [24], Theorem 5.1.11.

**3.1. Theorem.** (*Rokhlin's Disintegration Theorem*) Suppose that  $\Sigma$  is a complete and separable metric space,  $\Gamma$  is a measurable partition<sup>4</sup> of  $\Sigma$  and  $\mu$  is a probability on  $\Sigma$ . Then,  $\mu$  admits a disintegration relatively to  $\Gamma$ , i.e. a family  $\{\mu_\gamma\}_{\gamma \in \Gamma}$  of probabilities on  $\Sigma$  and a quotient measure  $\mu_x = \pi^* \mu$  such that, for all measurable set  $E \subset \Sigma$ :

- (a)  $\mu_\gamma(\gamma) = 1$  for  $\mu_x$ -a.e.  $\gamma \in \Gamma$ ;
- (b) the function  $\Gamma \rightarrow \mathbb{R}$ , defined by  $\gamma \mapsto \mu_\gamma(E)$  is measurable;
- (c)  $\mu(E) = \int \mu_\gamma(E) d\mu_x(\gamma)$ .

The proof of the following lemma can be found in [24], proposition 5.1.7.

**3.2. Lemma.** Suppose the  $\sigma$ -algebra  $\mathcal{B}$ , of  $\Sigma$ , has a countable generator. If  $(\{\mu_\gamma\}_{\gamma \in \Gamma}, \mu_x)$  and  $(\{\mu'_\gamma\}_{\gamma \in \Gamma}, \mu_x)$  are disintegrations of the measure  $\mu$  relatively to  $\Gamma$ , then  $\mu_\gamma = \mu'_\gamma$ ,  $\mu_x$ -almost every  $\gamma \in \Gamma$ .

Let  $(X, d)$  be a compact metric space,  $g : X \rightarrow \mathbb{R}$  be a Lipschitz function and let  $L(g)$  be its best Lipschitz constant, i.e.

$$L(g) = \sup_{x, y \in X} \left\{ \frac{|g(x) - g(y)|}{d(x, y)} \right\}.$$

**3.3. Definition.** Given two signed measures  $\mu$  and  $\nu$  on  $X$ , we define a **Wasserstein-Kantorovich Like** distance between  $\mu$  and  $\nu$  by

$$(8) \quad W_1^0(\mu, \nu) = \sup_{L(g) \leq 1, \|g\|_\infty \leq 1} \left| \int g d\mu - \int g d\nu \right|.$$

From now, we denote

$$(9) \quad \|\mu\|_W := W_1^0(0, \mu).$$

As a matter of fact,  $\|\cdot\|_W$  defines a norm on the vector space of signed measures defined on a compact metric space. We remark that this norm is equivalent the dual of the Lipschitz norm.

Let  $\mathcal{SB}(\Sigma)$  be the space of Borel signed measures on  $\Sigma$ . Given  $\mu \in \mathcal{SB}(\Sigma)$  denote by  $\mu^+$  and  $\mu^-$  the positive and the negative parts of it ( $\mu = \mu^+ - \mu^-$ ). Denote by  $\mathcal{AB}$  the set of signed measures  $\mu \in \mathcal{SB}(\Sigma)$  such that its associated positive and negative marginal measures,  $\pi_x^* \mu^+$  and  $\pi_x^* \mu^-$  are absolutely continuous with respect to the volume measure  $m_1$ , i.e.

$$(10) \quad \mathcal{AB} = \{\mu \in \mathcal{SB}(\Sigma) : \pi_x^* \mu^+ \ll m_1 \text{ and } \pi_x^* \mu^- \ll m_1\},$$

where  $\pi_x : \Sigma \rightarrow N_1$  is the projection defined by  $\pi(x, y) = x$ .

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<sup>4</sup>We say that a partition  $\Gamma$  is measurable if there exists a full measure set  $M_0 \subset \Sigma$  s.t. restricted to  $M_0$ ,  $\Gamma = \bigvee_{n=1}^\infty \Gamma_n$ , for some increasing sequence  $\Gamma_1 \prec \Gamma_2 \prec \dots \prec \Gamma_n \prec \dots$  of countable partitions of  $\Sigma$ . Furthermore,  $\Gamma_i \prec \Gamma_{i+1}$  means that each element of  $\mathcal{P}_{i+1}$  is a subset of some element of  $\Gamma_i$ .

Given a *probability measure*  $\mu \in \mathcal{AB}$  on  $\Sigma$ , theorem 3.1 describes a disintegration  $(\{\mu_\gamma\}_\gamma, \mu_x)$  along  $\mathcal{F}^s$  (see equation (2))<sup>5</sup> by a family  $\{\mu_\gamma\}_\gamma$  of probability measures on the stable leaves<sup>6</sup> and, since  $\mu \in \mathcal{AB}$ ,  $\mu_x$  can be identified with a non negative marginal density  $\phi_x : N_1 \rightarrow \mathbb{R}$ , defined almost everywhere, with  $|\phi_x|_1 = 1$ . For a positive measure  $\mu \in \mathcal{AB}$  we define its disintegration by disintegrating the normalization of  $\mu$ .

**3.4. Definition.** Let  $\pi_{\gamma,y} : \gamma \rightarrow N_2$  be the restriction  $\pi_y|_\gamma$ , where  $\pi_y : \Sigma \rightarrow N_2$  is the projection defined by  $\pi_y(x, y) = y$  and  $\gamma \in \mathcal{F}^s$ . Given a positive measure  $\mu \in \mathcal{AB}$  and its disintegration along the stable leaves  $\mathcal{F}^s$ ,  $(\{\mu_\gamma\}_\gamma, \mu_x = \phi_x m_1)$ , we define the **restriction of  $\mu$  on  $\gamma$**  as the positive measure  $\mu|_\gamma$  on  $N_2$  (not on the leaf  $\gamma$ ) defined, for all measurable set  $A \subset N_2$ , as

$$\mu|_\gamma(A) = \pi_{\gamma,y}^*(\phi_x(\gamma)\mu_\gamma)(A).$$

For a given signed measure  $\mu \in \mathcal{AB}$  and its decomposition  $\mu = \mu^+ - \mu^-$ , define the **restriction of  $\mu$  on  $\gamma$**  by

$$(12) \quad \mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma.$$

**3.5. Definition.** Let  $\mathcal{L}^1 \subseteq \mathcal{AB}$  be defined as

$$(13) \quad \mathcal{L}^1 = \left\{ \mu \in \mathcal{AB} : \int_{N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma) < \infty \right\}$$

and define a norm on it,  $\|\cdot\|_1 : \mathcal{L}^1 \rightarrow \mathbb{R}$ , by

$$(14) \quad \|\mu\|_1 = \int_{N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma).$$

Now, we define the following set of signed measures on  $\Sigma$ ,

$$(15) \quad S^1 = \{ \mu \in \mathcal{L}^1; \phi_x \in S_- \}.$$

Consider the function  $\|\cdot\|_{S^1} : S^1 \rightarrow \mathbb{R}$ , defined by

$$(16) \quad \|\mu\|_{S^1} = |\phi_x|_s + \|\mu\|_1,$$

where we denote  $\phi_x = \phi_x^+ - \phi_x^-$  with  $\phi_x^\pm$  being the marginals of  $\mu^\pm$  as explained before.  $\phi_x$  is the marginal density of the disintegration of  $\mu$  and we remark that  $\phi_x^+$  is not necessarily equal to the positive part of  $\phi_x$ .

The proof of the next proposition is straightforward. Details can be found in [23].

**3.6. Proposition.**  $(\mathcal{L}^1, \|\cdot\|_1)$  and  $(S^1, \|\cdot\|_{S^1})$  are normed vector spaces.

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<sup>5</sup>By lemma 3.2, the disintegration of a measure  $\mu$  is the  $\mu_x$ -unique measurable family  $(\{\mu_\gamma\}_\gamma, \phi_x m_1)$  such that, for every measurable set  $E \subset \Sigma$  it holds

$$(11) \quad \mu(E) = \int_{N_1} \mu_\gamma(E \cap \gamma) d(\phi_x m_1)(\gamma).$$

We also remark that, in our context,  $\Gamma$  and  $\pi$  of theorem 3.1 are respectively equal to  $\mathcal{F}^s$  and  $\pi_x$ , defined by  $\pi(x, y) = x$ , where  $x \in N_1$  and  $y \in N_2$ .

<sup>6</sup>In the following to simplify notations, when no confusion is possible we will indicate the generic leaf or its coordinate with  $\gamma$ .

### 3.2. $L^\infty$ like spaces.

**3.7. Definition.** Let  $\mathcal{L}^\infty \subseteq \mathcal{AB}(\Sigma)$  be defined as

$$(17) \quad \mathcal{L}^\infty = \{ \mu \in \mathcal{AB} : \text{ess sup}(W_1^0(\mu^+|_\gamma, \mu^-|_\gamma)) < \infty \},$$

where the essential supremum is taken over  $N_1$  with respect to  $m_1$ . Define the function  $\|\cdot\|_\infty : \mathcal{L}^\infty \rightarrow \mathbb{R}$  by

$$(18) \quad \|\mu\|_\infty = \text{ess sup}(W_1^0(\mu^+|_\gamma, \mu^-|_\gamma)).$$

Finally, consider the following set of signed measures on  $\Sigma$

$$(19) \quad S^\infty = \{ \mu \in \mathcal{L}^\infty; \phi_x \in S_- \},$$

and the function,  $\|\cdot\|_{S^\infty} : S^\infty \rightarrow \mathbb{R}$ , defined by

$$(20) \quad \|\mu\|_{S^\infty} = |\phi_x|_s + \|\mu\|_\infty.$$

The proof of the next proposition is straightforward and can be found in [23].

**3.8. Proposition.**  $(\mathcal{L}^\infty, \|\cdot\|_\infty)$  and  $(S^\infty, \|\cdot\|_{S^\infty})$  are normed vector spaces.

## 4. TRANSFER OPERATOR ASSOCIATED TO $F$

Let us now consider the transfer operator  $F^*$  associated with  $F$ , i.e. such that

$$[F^* \mu](E) = \mu(F^{-1}(E)),$$

for each signed measure  $\mu \in \mathcal{SB}(\Sigma)$  and for each measurable set  $E \subset \Sigma$ .

**4.1. Lemma.** For all probability  $\mu \in \mathcal{AB}$  disintegrated by  $(\{\mu_\gamma\}_\gamma, \phi_x)$ , the disintegration  $((F^* \mu)_\gamma, (F^* \mu)_x)$  of  $F^* \mu$  is given by

$$(21) \quad (F^* \mu)_x = P_T(\phi_x) m_1$$

and

$$(22) \quad (F^* \mu)_\gamma = \nu_\gamma := \frac{1}{P_T(\phi_x)(\gamma)} \sum_{i=1}^q \frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma) \cdot \chi_{T_i(P_i)}(\gamma) \cdot F^* \mu_{T_i^{-1}(\gamma)}$$

when  $P_T(\phi_x)(\gamma) \neq 0$ . Otherwise, if  $P_T(\phi_x)(\gamma) = 0$ , then  $\nu_\gamma$  is the Lebesgue measure on  $\gamma$  (the expression  $\frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma) \cdot \frac{\chi_{T_i(P_i)}(\gamma)}{P_T(\phi_x)(\gamma)} \cdot F^* \mu_{T_i^{-1}(\gamma)}$  is understood to be zero outside  $T_i(P_i)$  for all  $i = 1, \dots, q$ ). Here and above,  $\chi_A$  is the characteristic function of the set  $A$ .

*Proof.* By the uniqueness of the disintegration (see Lemma 3.2) to prove Lemma 4.1, is enough to prove the following equation

$$(23) \quad F^* \mu(E) = \int_{N_1} \nu_\gamma(E \cap \gamma) P_T(\phi_x)(\gamma) d\gamma,$$

for a measurable set  $E \subset \Sigma$ . To do it, let us define the sets  $B_1 = \{\gamma \in N_1; T^{-1}(\gamma) = \emptyset\}$ ,  $B_2 = \{\gamma \in B_1^c; P_T(\phi_x)(\gamma) = 0\}$  and  $B_3 = (B_1 \cup B_2)^c$ . The following properties can be easily proven:

1.  $B_i \cap B_j = \emptyset$ ,  $T^{-1}(B_i) \cap T^{-1}(B_j) = \emptyset$ , for all  $1 \leq i, j \leq 3$  such that  $i \neq j$   
and  $\bigcup_{i=1}^3 B_i = \bigcup_{i=1}^3 T^{-1}(B_i) = N_1$ ;
2.  $m_1(T^{-1}(B_1)) = m_1(T^{-1}(B_2)) = 0$ ;

Using the change of variables  $\gamma = T_i(\beta)$  and the definition of  $\nu_\gamma$  (see (22)), we have

$$\begin{aligned}
\int_{N_1} \nu_\gamma(E \cap \gamma) P_T(\phi_x)(\gamma) d\gamma &= \int_{B_3} \sum_{i=1}^q \frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma) F^* \mu_{T_i^{-1}(\gamma)}(E) \chi_{T_i(P_i)(\gamma)} dm_1(\gamma) \\
&= \sum_{i=1}^q \int_{T_i(P_i) \cap B_3} \frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma) F^* \mu_{T_i^{-1}(\gamma)}(E) dm_1(\gamma) \\
&= \sum_{i=1}^q \int_{P_i \cap T_i^{-1}(B_3)} \phi_x(\beta) \mu_\beta(F^{-1}(E)) dm_1(\beta) \\
&= \int_{T^{-1}(B_3)} \phi_x(\beta) \mu_\beta(F^{-1}(E)) dm_1(\beta) \\
&= \int_{\bigcup_{i=1}^3 T^{-1}(B_i)} \mu_\beta(F^{-1}(E)) d\phi_x m_1(\beta) \\
&= \int_{N_1} \mu_\beta(F^{-1}(E)) d\phi_x m_1(\beta) \\
&= \mu(F^{-1}(E)) \\
&= F^* \mu(E).
\end{aligned}$$

And the proof is done.  $\square$

Now, if  $\mu \in \mathcal{L}^1$ , applying the above Lemma to  $\mu^+$  and  $\mu^-$  we directly get

**4.2. Proposition.** *Let  $\gamma \in \mathcal{F}^s$  be a stable leaf. Let us define the map  $F_\gamma : N_2 \rightarrow N_2$  by*

$$F_\gamma = \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1}.$$

*Then, for each  $\mu \in \mathcal{L}^1$  and for almost all  $\gamma \in N_1$  (interpreted as the quotient space of leaves) it holds*

$$(24) \quad (F^* \mu)|_\gamma = \sum_{i=1}^q \frac{F_{T_i^{-1}(\gamma)}^* \mu|_{T_i^{-1}(\gamma)}}{|\det DT_i \circ T_i^{-1}(\gamma)|} \chi_{T_i(P_i)(\gamma)} \quad \text{for almost all } \gamma \in N_1.$$

## 5. BASIC PROPERTIES OF THE NORMS AND CONVERGENCE TO EQUILIBRIUM

In this section, we show important properties of the norms and their behavior with respect to the transfer operator. In particular, we show that the  $\mathcal{L}^1$  norm is weakly contracted by the transfer operator. We prove Lasota-Yorke like inequalities for the strong norms and exponential convergence to equilibrium statements. All these properties will be used in next section to prove a spectral gap statement for the transfer operator.

**5.1. Proposition** (The weak norm is weakly contracted by  $F^*$ ). *If  $\mu \in \mathcal{L}^1$  then*

$$(25) \quad \|F^* \mu\|_1 \leq \|\mu\|_1.$$

In the proof of the proposition we will use the following lemma about the behavior of the  $\|\cdot\|_W$  norm (see equation (9)) after a contraction. Essentially it says that a contraction cannot increase the  $\|\cdot\|_W$  norm.



**5.2. Lemma.** *For every  $\mu \in \mathcal{AB}$  and a stable leaf  $\gamma \in \mathcal{F}^s$ , it holds*

$$(26) \quad \|F_\gamma^* \mu|_\gamma\|_W \leq \|\mu|_\gamma\|_W,$$

where  $F_\gamma : N_2 \rightarrow N_2$  is defined in Proposition 4.2. Moreover, if  $\mu$  is a probability measure

$$(27) \quad \|F^{*n} \mu\|_W = \|\mu\|_W = 1, \quad \forall n \geq 1.$$

*Proof.* (of Lemma 5.2) Indeed, since  $F_\gamma$  is an  $\alpha$ -contraction, if  $|g|_\infty \leq 1$  and  $Lip(g) \leq 1$  the same holds for  $g \circ F_\gamma$ . Since

$$\left| \int g dF_\gamma^* \mu \right| = \left| \int g(F_\gamma) d\mu \right|,$$

taking the supremum over  $|g|_\infty \leq 1$  and  $Lip(g) \leq 1$  we finish the proof of the inequality. Equation (27) is trivial since if  $\mu$  is a probability measure.  $\square$

Now we are ready to prove Proposition 5.1.

*Proof.* (of Proposition 5.1 )

In the following we consider, for all  $i$ , the change of variable  $\gamma = T_i(\alpha)$ . Thus, Lemma 5.2 and equation (24) yield

$$\begin{aligned} \|F^* \mu\|_1 &= \int_{N_1} \|(F^* \mu)|_\gamma\|_W dm_1(\gamma) \\ &\leq \sum_{i=1}^q \int_{T(P_i)} \left\| \frac{F_{T_i^{-1}(\gamma)}^* \mu|_{T_i^{-1}(\gamma)}}{|\det DT_i(T_i^{-1}(\gamma))|} \right\|_W dm_1(\gamma) \\ &= \sum_{i=1}^q \int_{P_i} \|F_\alpha^* \mu|_\alpha\|_W dm_1(\alpha) \\ &= \sum_{i=1}^q \int_{P_i} \|\mu|_\alpha\|_W dm_1(\alpha) \\ &= \|\mu\|_1. \end{aligned}$$

$\square$

The following proposition shows a regularizing action of the transfer operator with respect to the strong norm. Such inequalities are usually called Lasota-Yorke or Doeblin-Fortet inequalities.

**5.3. Proposition** (Lasota-Yorke inequality for  $S^1$ ). *There exist  $A, B_2 \in \mathbb{R}, \lambda < 1$  such that, for all  $\mu \in S^1$ , it holds*

$$(28) \quad \|F^{*n} \mu\|_{S^1} \leq A\lambda^n \|\mu\|_{S^1} + B_2 \|\mu\|_1 \quad \forall n \geq 1.$$

Before the proof of the proposition we prove a preliminary Lemma

**5.4. Lemma.** *Let  $k, \beta_0$  and  $C$  be the constants of assumption **T3.3**, then there is  $\overline{C} > 0$  such that for all  $\mu \in S^1$ , it holds*

$$(29) \quad \|F^{*k} \mu\|_{S^1} \leq \beta_0 \|\mu\|_{S^1} + \overline{C} \|\mu\|_1.$$

*Proof.* (of Lemma 5.4 ) Firstly, we recall that  $\phi_x$  is the marginal density of the disintegration of  $\mu$ . Precisely,  $\phi_x = \phi_x^+ - \phi_x^-$ , where  $\phi_x^+ = \frac{d\pi_x^* \mu^+}{dm_1}$  and  $\phi_x^- = \frac{d\pi_x^* \mu^-}{dm_1}$ . Set  $\overline{C} = 1 + C$ . Thus, it holds (note that  $|\phi_x|_1 \leq \|\mu\|_1$ )

$$\begin{aligned} \|\mathbf{F}^{*k} \mu\|_{S^1} &= |\mathbf{P}_T^k \phi_x|_s + \|\mathbf{F}^{*k} \mu\|_1 \\ &\leq \beta_0 |\phi_x|_s + C |\phi_x|_1 + \|\mu\|_1 \\ &\leq \beta_0 (|\phi_x|_s + \|\mu\|_1) + C \|\mu\|_1 + \|\mu\|_1 \\ &\leq \beta_0 \|\mu\|_{S^1} + \overline{C} \|\mu\|_1. \end{aligned}$$

□

*Proof.* (of Proposition 5.3 )

Note that, iterating one time the inequality (29), we get

$$\|\mathbf{F}^{*2k} \mu\|_{S^1} \leq \beta_0^2 \|\mu\|_{S^1} + \overline{C}(1 + \beta_0) \|\mu\|_1.$$

Thus, for all  $s \in \mathbb{N}$ , we have

$$\begin{aligned} \|\mathbf{F}^{*sk} \mu\|_{S^1} &\leq \beta_0^s \|\mu\|_{S^1} + \overline{C}(1 + \beta_0 + \dots + \beta_0^{s-1}) \|\mu\|_1 \\ &\leq \beta_0^s \|\mu\|_{S^1} + \frac{\overline{C}}{1 - \beta_0} \|\mu\|_1. \end{aligned}$$

Therefore, for all  $s \in \mathbb{R}$ , it holds

$$(30) \quad \|\mathbf{F}^{*sk} \mu\|_{S^1} \leq \beta_0^s \|\mu\|_{S^1} + \frac{\overline{C}}{1 - \beta_0} \|\mu\|_1.$$

For a given  $n \in \mathbb{N}$ , let  $n = q_n k + r_n$ , where  $0 \leq r_n \leq k$ . Since  $\mathbf{P}_T : S_- \rightarrow S_-$  is bounded, there exists  $M_1 > 0$  such that  $|\mathbf{P}_T^{r_n}|_s \leq M_1$  for all  $n$ , where  $|\mathbf{P}_T^{r_n}|_s = \sup_{f \in S_-, f \neq 0} \frac{|\mathbf{P}_T^{r_n}(f)|_s}{|f|_s}$ . Thus,

$$\begin{aligned} \|\mathbf{F}^{*n} \mu\|_{S^1} &\leq \beta_0^{q_n} \|\mathbf{F}^{*r_n} \mu\|_{S^1} + \frac{\overline{C}}{1 - \beta_0} \|\mu\|_1 \\ &\leq \beta_0^{q_n} (|\mathbf{P}_T^{r_n}(\phi_x)|_s + \|\mathbf{F}^{*r_n} \mu\|_1) + \frac{\overline{C}}{1 - \beta_0} \|\mu\|_1 \\ &\leq \beta_0^{q_n} (M_1 |\phi_x|_s + \|\mu\|_1) + \frac{\overline{C}}{1 - \beta_0} \|\mu\|_1 \\ &\leq \left(\beta_0^{\frac{1}{k}}\right)^n \left(\frac{1}{\beta_0}\right)^{\frac{r_n}{k}} M_1 \|\mu\|_S + \left(1 + \frac{\overline{C}}{1 - \beta_0}\right) \|\mu\|_1 \\ &\leq \lambda^n A \|\mu\|_S + B_2 \|\mu\|_1, \end{aligned}$$

where  $\lambda = \beta_0^{\frac{1}{k}}$ ,  $A = \frac{M_1}{\beta_0}$  and  $B_2 = 1 + \frac{\overline{C}}{1 - \beta_0}$ .

□

**5.1. Convergence to equilibrium.** In general, we say that the a transfer operator  $L$  has convergence to equilibrium with at least speed  $\Phi$  and with respect to norms  $\|\cdot\|_s$  and  $\|\cdot\|_w$ , if for each  $f \in \mathcal{V}_s = \{f \in B_s, f(X) = 0\}$ , it holds

$$(31) \quad \|L^n f\|_w \leq \Phi(n)\|f\|_s,$$

where  $\Phi(n) \rightarrow 0$  as  $n \rightarrow \infty$ .

In this chapter, we prove that  $F$  has exponential convergence to equilibrium. This is weaker with respect to spectral gap. However, the spectral gap follows from the above Lasota-Yorke inequality and the convergence to equilibrium. To do it, we need some preliminary lemma and the following is somewhat similar to Lemma 5.2 considering the behaviour of the  $\|\cdot\|_W$  norm after a contraction. It gives a finer estimate for zero average measures.

**5.5. Lemma.** *For all signed measures  $\mu$  on  $N_2$  and for all  $\gamma \in N_1 (= \mathcal{F}^s)$ , it holds*

$$\|F_\gamma^* \mu\|_W \leq \alpha \|\mu\|_W + \mu(N_2)$$

( $\alpha$  is the rate of contraction of  $G$ ). In particular, if  $\mu(N_2) = 0$  then

$$\|F_\gamma^* \mu\|_W \leq \alpha \|\mu\|_W.$$

*Proof.* If  $Lip(g) \leq 1$  and  $\|g\|_\infty \leq 1$ , then  $g \circ F_\gamma$  is  $\alpha$ -Lipschitz. Moreover, since  $\|g\|_\infty \leq 1$ , then  $\|g \circ F_\gamma - \theta\|_\infty \leq \alpha$ , for some  $\theta \leq 1$ . Indeed, let  $z \in N_2$  be such that  $|g \circ F_\gamma(z)| \leq 1$ , set  $\theta = g \circ F_\gamma(z)$  and let  $d_2$  be the Riemannian metric of  $N_2$ . Since  $\text{diam}(N_2) = 1$ , we have

$$|g \circ F_\gamma(y) - \theta| \leq \alpha d_2(y, z) \leq \alpha$$

and consequently  $\|g \circ F_\gamma - \theta\|_\infty \leq \alpha$ .

This implies,

$$\begin{aligned} \left| \int_{N_2} g dF_\gamma^* \mu \right| &= \left| \int_{N_2} g \circ F_\gamma d\mu \right| \\ &\leq \left| \int_{N_2} g \circ F_\gamma - \theta d\mu \right| + \left| \int_{N_2} \theta d\mu \right| \\ &= \alpha \left| \int_{N_2} \frac{g \circ F_\gamma - \theta}{\alpha} d\mu \right| + \theta |\mu(N_2)|. \end{aligned}$$

And taking the supremum over  $|g|_\infty \leq 1$  and  $Lip(g) \leq 1$ , we have  $\|F_\gamma^* \mu\|_W \leq \alpha \|\mu\|_W + \mu(N_2)$ . In particular, if  $\mu(N_2) = 0$ , we get the second part.  $\square$

**5.6. Proposition.** *For all signed measure  $\mu \in \mathcal{L}^1$ , it holds*

$$(32) \quad \|F^* \mu\|_1 \leq \alpha \|\mu\|_1 + (\alpha + 1)|\phi_x|_1.$$

*Proof.* Consider a signed measure  $\mu \in \mathcal{L}^1$  and its restriction on the leaf  $\gamma$ ,  $\mu|_\gamma = \pi_{\gamma,y}^*(\phi_x(\gamma)\mu_\gamma)$ . Set

$$\bar{\mu}|_\gamma = \pi_{\gamma,y}^* \mu_\gamma.$$

If  $\mu$  is a positive measure then  $\bar{\mu}|_\gamma$  is a probability on  $N_2$  and  $\mu|_\gamma = \phi_x(\gamma)\bar{\mu}|_\gamma$ . Then, the expression given by Proposition 4.2 yields

$$\begin{aligned}
\|F^* \mu\|_1 &\leq \sum_{i=1}^q \int_{T(P_i)} \left\| \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^+(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} - \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^-}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} \right\|_W dm_1(\gamma) \\
&\leq \sum_{i=1}^q \int_{T(P_i)} \left\| \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^+(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} - \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} \right\|_W dm_1(\gamma) \\
&\quad + \sum_{i=1}^q \int_{T(P_i)} \left\| \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} - \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^-}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} \right\|_W dm_1(\gamma) \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \sum_{i=1}^q \int_{T(P_i)} \left\| \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^+(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} - \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} \right\|_W dm_1(\gamma)$$

and

$$I_2 = \sum_{i=1}^q \int_{T(P_i)} \left\| \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} - \frac{F_{T_i^{-1}(\gamma)}^* \overline{\mu^-}|_{T_i^{-1}(\gamma)} \phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i| \circ T_i^{-1}(\gamma)} \right\|_W dm_1(\gamma).$$

Let us estimate  $I_1$  and  $I_2$ .

By Lemma 5.2 and a change of variable we have

$$\begin{aligned}
I_1 &= \sum_{i=1}^q \int_{T(P_i)} \left\| F_{T_i^{-1}(\gamma)}^* \overline{\mu^+}|_{T_i^{-1}(\gamma)} \right\|_W \frac{|\phi_x^+ - \phi_x^-|}{|\det DT_i|} \circ T_i^{-1}(\gamma) dm_1(\gamma) \\
&\leq \int_{N_1} \left\| F_\beta^* \overline{\mu^+}|_\beta \right\|_W |\phi_x^+ - \phi_x^-|(\beta) dm_1(\beta) \\
&= \int_{N_1} |\phi_x^+ - \phi_x^-|(\beta) dm_1(\beta) \\
&= |\phi_x|_1,
\end{aligned}$$

and by Lemma 5.5 we have

$$\begin{aligned}
I_2 &= \sum_{i=1}^q \int_{T(P_i)} \left\| F_{T_i^{-1}(\gamma)}^* \left( \overline{\mu^+}|_{T_i^{-1}(\gamma)} - \overline{\mu^-}|_{T_i^{-1}(\gamma)} \right) \right\|_W \frac{\phi_x^-}{|\det DT_i|} \circ T_i^{-1}(\gamma) dm_1(\gamma) \\
&\leq \sum_{i=1}^q \int_{P_i} \left\| F_{\beta}^* \left( \overline{\mu^+}|_{\beta} - \overline{\mu^-}|_{\beta} \right) \right\|_W \phi_x^-(\beta) dm_1(\beta) \\
&\leq \alpha \int_{N_1} \left\| \overline{\mu^+}|_{\beta} - \overline{\mu^-}|_{\beta} \right\|_W \phi_x^-(\beta) dm_1(\beta) \\
&\leq \alpha \int_{N_1} \left\| \overline{\mu^+}|_{\beta} \phi_x^-(\beta) - \overline{\mu^+}|_{\beta} \phi_x^+(\beta) \right\|_W dm_1(\beta) \\
&\leq \alpha \int_{N_1} \left\| \overline{\mu^+}|_{\beta} \phi_x^-(\beta) - \overline{\mu^+}|_{\beta} \phi_x^+(\beta) \right\|_W dm_1(\beta) + \alpha \int_{N_1} \left\| \overline{\mu^+}|_{\beta} \phi_x^+(\beta) - \overline{\mu^-}|_{\beta} \phi_x^-(\beta) \right\|_W dm_1(\beta) \\
&= \alpha |\phi_x|_1 + \alpha \|\mu\|_1.
\end{aligned}$$

Summing the above estimates we finish the proof.  $\square$

Iterating (32) we get the following corollary.

**5.7. Corollary.** *For all signed measure  $\mu \in \mathcal{L}^1$  it holds*

$$\|F^{*n} \mu\|_1 \leq \alpha^n \|\mu\|_1 + \overline{\alpha} |\phi_x|_1,$$

where  $\overline{\alpha} = \frac{1+\alpha}{1-\alpha}$ .

Let us consider the set of zero average measures in  $S^1$  defined by

$$(33) \quad \mathcal{V}_s = \{\mu \in S^1 : \mu(\Sigma) = 0\}.$$

Note that, for all  $\mu \in \mathcal{V}_s$  we have  $\pi_x^* \mu(N_1) = 0$ . Moreover, since  $\pi_x^* \mu = \phi_x m_1$  ( $\phi_x = \phi_x^+ - \phi_x^-$ ), we have  $\int_{N_1} \phi_x dm_1 = 0$ . This allows us to apply Theorem 2.1 in the proof of the next proposition.

**5.8. Proposition** (Exponential convergence to equilibrium). *There exist  $D_2 \in \mathbb{R}$  and  $0 < \beta_1 < 1$  such that, for every signed measure  $\mu \in \mathcal{V}_s$ , it holds*

$$\|F^{*n} \mu\|_1 \leq D_2 \beta_1^n \|\mu\|_{S^1},$$

for all  $n \geq 1$ .

*Proof.* Given  $\mu \in \mathcal{V}_s$  and denoting  $\phi_x = \phi_x^+ - \phi_x^-$ , it holds that  $\int \phi_x dm_1 = 0$ . Moreover, Theorem 2.1 yields  $|\mathbf{P}_T^n(\phi_x)|_s \leq Dr^n |\phi_x|_s$  for all  $n \geq 1$ , then  $|\mathbf{P}_T^n(\phi_x)|_s \leq Dr^n \|\mu\|_{S^1}$  for all  $n \geq 1$ .

Let  $l$  and  $0 \leq d \leq 1$  be the coefficients of the division of  $n$  by 2, i.e.  $n = 2l + d$ . Thus,  $l = \frac{n-d}{2}$  (by Proposition 5.1, we have  $\|F^{*s} \mu\|_1 \leq \|\mu\|_1$ , for all  $s$ , and  $\|\mu\|_1 \leq \|\mu\|_{S^1}$ ) and by Corollary 5.7, it holds (below, set  $\beta_1 = \sup\{\sqrt{r}, \sqrt{\alpha}\}$ )

$$\begin{aligned}
\|F^{*n}\mu\|_1 &\leq \|F^{*2l+d}\mu\|_1 \\
&\leq \alpha^l \|F^{*l+d}\mu\|_1 + \bar{\alpha} \left| \frac{d(\pi_x^*(F^{*l+d}\mu))}{dm_1} \right|_1 \\
&\leq \alpha^l \|\mu\|_1 + \bar{\alpha} |P_T^l(\phi_x)|_1 \\
&\leq (1 + \bar{\alpha}D)\beta_1^{-d}\beta_1^n \|\mu\|_{S^1} \\
&\leq D_2\beta_1^n \|\mu\|_{S^1},
\end{aligned}$$

where  $D_2 = \frac{1 + \bar{\alpha}D}{\beta_1}$ . □

**5.9. Remark.** We remark that the rate of convergence to equilibrium,  $\beta_1$ , for the map  $F$  found above, is directly related to the rate of contraction,  $\alpha$ , of the stable foliation, and on the rate of convergence to equilibrium,  $r$ , of the induced basis map  $T$  (see equation 5). More precisely,  $\beta_1 = \max\{\sqrt{\alpha}, \sqrt{r}\}$ . Similarly, we have an explicit estimate for the constant  $D_2$ , provided we have an estimate for  $D$  in the basis map<sup>7</sup>.

Now recall we denoted by  $\psi_x$  the unique  $T$ -invariant density in  $S_-$  (see T3.4). Following the construction exposed in [29] (subsection 7.3.4.1) we consider  $\mu_0$  as the  $F$ -invariant measure such that  $\frac{d(\pi_x^*\mu_0)}{dm_1} = \psi_x \in S_-$ . This motivates the following proposition.

**5.10. Proposition.** *The unique invariant measure for the system  $F : N_1 \times N_2 \rightarrow N_1 \times N_2$  in  $S^1$  is  $\mu_0$ . Moreover, if **N1** is satisfied,  $\mu_0$  is the unique  $F$ -invariant measure in  $S^\infty$ .*

*Proof.* Let  $\mu_0$  be the  $F$ -invariant measure such that  $\frac{d(\pi_x^*\mu_0)}{dm_1} = \psi_x \in S_-$ , where  $\psi_x$  is the unique  $T$ -invariant density (see T3.4) in  $S_-$ . Define the probability  $\bar{\mu}_0|_\gamma = \pi_y^*\mu_0|_\gamma$ . Since  $\|\bar{\mu}_0|_\gamma\|_W = 1$  (it is a probability), we have  $\|\mu_0|_\gamma\|_W = |\psi_x(\gamma)|\|\bar{\mu}_0|_\gamma\|_W = |\psi_x(\gamma)|$ . So  $\int \|\mu_0|_\gamma\|_W dm_1(\gamma) = \int |\psi_x(\gamma)| dm_1(\gamma) = |\psi_x|_1 < \infty$ . Then  $\mu_0 \in \mathcal{L}^1$ . By construction,  $\psi_x \in S_-$ . Then  $\mu_0 \in S^1$ . And we are done.

If **N1** is satisfied, we have  $|\cdot|_\infty \leq |\cdot|_s$ . Suppose that  $g : N_2 \rightarrow \mathbb{R}$  is a Lipschitz function such that  $|g|_\infty \leq 1$  and  $L(g) \leq 1$ . Then, it holds  $|\int g d(\mu_0|_\gamma)| \leq |g|_\infty \psi_x(\gamma) \leq |\psi_x|_\infty \leq |\psi_x|_s$ . Hence,  $\mu_0 \in S^\infty$ .

For the uniqueness, if  $\mu_0, \mu_1 \in S^1$  are  $F$ -invariant, then  $\mu_0 - \mu_1 \in \mathcal{V}_s$ . By Proposition 5.8,  $F^{*n}(\mu_0 - \mu_1) \rightarrow 0$  in  $\mathcal{L}^1$ . Therefore,  $\mu_0 - \mu_1 = 0$ . □

**5.2.  $L^\infty$  norms.** In this section we consider an  $L^\infty$  like anisotropic norm. We show how a Lasota Yorke inequality can be proved for this norm too.

**5.11. Lemma.** *Under the assumptions G1, T1, ..., T3.3, for all signed measure  $\mu \in S^\infty$  with marginal density  $\phi_x$  it holds*

$$\|F^*\mu\|_\infty \leq \alpha |P_T 1|_\infty \|\mu\|_\infty + |P_T \phi_x|_\infty.$$

---

<sup>7</sup>It can be difficult to find a sharp estimate for  $D$ . An approach allowing to find some useful upper estimates is shown in [15]

*Proof.* Let  $T_i$  be the branches of  $T$ , for all  $i = 1 \cdots q$ . Applying Lemma 5.5 on the third line below, we have

$$\begin{aligned}
\|(F^* \mu)|_\gamma\|_W &= \left\| \sum_{i=1}^q \frac{F_{T_i^{-1}(\gamma)}^* \mu|_{T_i^{-1}(\gamma)}}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma) \right\|_W \\
&\leq \sum_{i=1}^q \frac{\|F_{T_i^{-1}(\gamma)}^* \mu|_{T_i^{-1}(\gamma)}\|_W}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma) \\
&\leq \sum_{i=1}^q \frac{\alpha \|\mu|_{T_i^{-1}(\gamma)}\|_W + \phi_x(T_i^{-1}(\gamma))}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma) \\
&\leq \alpha \|\mu\|_\infty \sum_{i=1}^q \frac{\chi_{T(P_i)}(\gamma)}{|\det DT_i(T_i^{-1}(\gamma))|} + \sum_{i=1}^q \frac{\phi_x(T_i^{-1}(\gamma))}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma).
\end{aligned}$$

Hence, taking the supremum on  $\gamma$ , we finish the proof of the statement.  $\square$

Applying the last lemma to  $F^{*n}$  instead of  $F$  one obtains.

**5.12. Lemma.** *Under the assumptions G1, T1, ..., T3.4, for all signed measure  $\mu \in S^\infty$  it holds*

$$\|F^{*n} \mu\|_\infty \leq \alpha^n |P_T^n 1|_\infty \|\mu\|_\infty + |P_T^n \phi_x|_\infty,$$

where  $\phi_x$  is the marginal density of  $\mu$ .

**5.13. Proposition** (Lasota-Yorke inequality for  $S^\infty$ ). *Suppose  $F$  satisfies the assumptions G1, T1, ..., T3.4 and N1. Then, there are  $0 < \alpha_1 < 1$  and  $A_1, B_4 \in \mathbb{R}$  such that for all  $\mu \in S^\infty$ , it holds*

$$\|F^{*n} \mu\|_{S^\infty} \leq A_1 \alpha_1^n \|\mu\|_{S^\infty} + B_4 \|\mu\|_1.$$

*Proof.* We remark that, by equation (7) and (N1) it follows  $|P_T^n 1|_\infty \leq H_N(B_3 + C_2)$ , for each  $n$ . Then,

$$\begin{aligned}
\|F^{*n} \mu\|_{S^\infty} &= |P_T^n \phi_x|_s + \|F^{*n} \mu\|_\infty \\
&\leq [B_3 \beta_2^n |\phi_x|_s + C_2 |\phi_x|_1] + [\alpha^n |P_T^n 1|_\infty \|\mu\|_\infty + |P_T^n \phi_x|_\infty] \\
&\leq [B_3 \beta_2^n |\phi_x|_s + C_2 |\phi_x|_1] \\
&\quad + [\alpha^n H_N(B_3 + C_2) \|\mu\|_\infty + H_N(B_3 \beta_2^n |\phi_x|_s + C_2 |\phi_x|_1)]. \\
&\leq [\max(\alpha, \beta_2)]^n [B_3(1 + 2H_N) + H_N C_2] \|\mu\|_{S^\infty} + C_2(1 + H_N) \|\mu\|_1,
\end{aligned}$$

where  $|\phi_x|_1 \leq \|\mu\|_1$  and  $|\phi_x|_s \leq \|\mu\|_{S^\infty}$ . We finish the proof, setting  $\alpha_1 = \max(\alpha, \beta_2)$ ,  $A_1 = [B_3(1 + 2H_N) + H_N C_2]$  and  $B_4 = C_2(1 + H_N)$ .  $\square$

## 6. SPECTRAL GAP

In this section, we prove a spectral gap statement for the transfer operator applied to our strong spaces. For this, we will directly use the properties proved in the previous section, and this will give a kind of constructive proof. We remark that, we cannot apply the traditional Hennion, or Ionescu-Tulcea and Marinescu's approach to our function spaces because there is no compact immersion of the strong space into the weak one. This comes from the fact that we are considering the same "dual of Lipschitz" distance in the contracting direction for both spaces.

**6.1. Theorem** (Spectral gap on  $S^1$ ). *If  $F$  satisfies **G1**, **T1**, ..., **T3.4** given at beginning of section 2, then the operator  $F^* : S^1 \rightarrow S^1$  can be written as*

$$F^* = P + N,$$

where

- a)  $P$  is a projection i.e.  $P^2 = P$  and  $\dim \text{Im}(P) = 1$ ;
- b) there are  $0 < \xi < 1$  and  $K > 0$  such that <sup>8</sup>  $\forall \mu \in S^1$

$$\|N^n(\mu)\|_{S^1} \leq \|\mu\|_{S^1} \xi^n K;$$

- c)  $PN = NP = 0$ .

*Proof.* First, let us show there exist  $0 < \xi < 1$  and  $K_1 > 0$  such that, for all  $n \geq 1$ , it holds

$$(34) \quad \|F^{*n}\|_{\mathcal{V}_s \rightarrow \mathcal{V}_s} \leq \xi^n K_1.$$

Indeed, consider  $\mu \in \mathcal{V}_s$  (see equation (33)) s.t.  $\|\mu\|_{S^1} \leq 1$  and for a given  $n \in \mathbb{N}$  let  $m$  and  $0 \leq d \leq 1$  be the coefficients of the division of  $n$  by 2, i.e.  $n = 2m + d$ . Thus  $m = \frac{n-d}{2}$ . By the Lasota-Yorke inequality (Proposition 5.3) we have the uniform bound  $\|F^{*n}\mu\|_{S^1} \leq B_2 + A$  for all  $n \geq 1$ . Moreover, by Propositions 5.8 and 5.1 there is some  $D_2$  such that it holds (below, let  $\lambda_0$  be defined by  $\lambda_0 = \max\{\beta_1, \lambda\}$ )

$$\begin{aligned} \|F^{*n}\mu\|_{S^1} &\leq A\lambda^m \|F^{*m+d}\mu\|_{S^1} + B_2 \|F^{*m+d}\mu\|_1 \\ &\leq \lambda^m A(A + B_2) + B_2 \|F^{*m}\mu\|_1 \\ &\leq \lambda^m A(A + B_2) + B_2 D_2 \beta_1^m \\ &\leq \lambda_0^m [A(A + B_2) + B_2 D_2] \\ &\leq \lambda_0^{\frac{n-d}{2}} [A(A + B_2) + B_2 D_2] \\ &\leq \left(\sqrt{\lambda_0}\right)^n \left(\frac{1}{\lambda_0}\right)^{\frac{d}{2}} [A(A + B_2) + B_2 D_2] \\ &= \xi^n K_1, \end{aligned}$$

where  $\xi = \sqrt{\lambda_0}$  and  $K_1 = \left(\frac{1}{\lambda_0}\right)^{\frac{1}{2}} [A(A + B_2) + B_2 D_2]$ . Thus, we arrive at

$$(35) \quad \|(F^*|_{\mathcal{V}})^n\|_{S^1 \rightarrow S^1} \leq \xi^n K_1.$$

Now, recall that  $F^* : S^1 \rightarrow S^1$  has an unique fixed point  $\mu_0$ . Consider the operator  $P : S^1 \rightarrow [\mu_0]$  ( $[\mu_0]$  is the space spanned by  $\mu_0$ ), defined by  $P(\mu) = \mu(\Sigma)\mu_0$ . By definition  $P$  is a projection. Define the operator

$$S : S^1 \rightarrow \mathcal{V}_s,$$

by

$$S(\mu) = \mu - P(\mu) \quad \forall \mu \in S^1.$$

Thus, we set  $N = F^* \circ S$  and observe that, by definition,  $PN = NP = 0$  and  $F^* = P + N$ . Moreover,  $N^n(\mu) = F^{*n}(S(\mu))$  for all  $n \geq 1$ . Since  $S$  is bounded and  $S(\mu) \in \mathcal{V}_s$  we get, by (35),  $\|N^n(\mu)\|_{S^1} \leq \xi^n K \|\mu\|_{S^1}$ , for all  $n \geq 1$ , where  $K = K_1 \|S\|_{S^1 \rightarrow S^1}$ .  $\square$

<sup>8</sup>We remark that, by this reason, the spectral radius of  $\overline{N}$  satisfies  $\rho(\overline{N}) < 1$ , where  $\overline{N}$  is the extension of  $N$  to  $\overline{S^1}$  (the completion of  $S^1$ ). This gives us spectral gap, in the usual sense, for the operator  $\overline{F} : \overline{S^1} \rightarrow \overline{S^1}$ . The same remark holds for Theorem ??.



In the same way, using the  $\mathcal{L}^\infty$  Lasota Yorke inequality of Proposition 5.13, it is possible to obtain spectral gap on the  $L^\infty$  like space, we omit the proof which is essentially the same as above:

**6.2. Theorem** (Spectral gap on  $S^\infty$ ). *If  $F$  satisfies the assumptions G1, T1, ..., T3.4 and N1, then the operator  $F^* : S^\infty \rightarrow S^\infty$  can be written as*

$$F^* = P + N,$$

where

- a)  $P$  is a projection i.e.  $P^2 = P$  and  $\dim \text{Im}(P) = 1$ ;
- b) there are  $0 < \xi_1 < 1$  and  $K_2 > 0$  such that  $\|N^n(\mu)\|_{S^\infty} \leq \|\mu\|_{S^\infty} \xi_1^n K_2$   
 $\forall \mu \in S^\infty$ ;
- c)  $P N = N P = 0$ .

**6.3. Remark.** We remark the "gap",  $\xi$ , for the map  $F$  found in Theorem 6.1, is directly related to the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of  $F$  found before (see Remark 5.9). More precisely,  $\xi = \sqrt{\max\{\lambda, \beta_1\}}$ . We remark that, from the above proof we also have an explicit estimate for  $K$  in the exponential convergence, while many classical approaches are not suitable for this.

## 7. APPLICATION TO LORENZ-LIKE MAPS

In this section, we apply Theorems 6.1 and 6.2 to a large class of maps which are Poincaré maps for suitable sections of Lorenz-like flows. In these systems (see e.g [4]), it can be proved that there is a two dimensional Poincaré section  $\Sigma$  which can be supposed to be a rectangle, whose return map  $F_L : [0, 1]^2 \rightarrow [0, 1]^2$ , after a suitable change of coordinates, has the form  $F_L(x, y) = (T_L(x), G_L(x, y))$ , having all properties given at beginning of section 2. The map  $T_L : [0, 1] \rightarrow [0, 1]$ , in this case, can be supposed to be piecewise expanding with  $C^{1+\alpha}$  branches. Hence we consider a class of skew product maps  $[0, 1]^2 \rightarrow [0, 1]^2$  satisfying (G1), (T1), (T2), and the following properties on  $T_L$  :

7.0.1. *Properties of  $T_L$  in Lorenz-like systems .*

(P'1)  $\frac{1}{|T'_L|}$  is of universal bounded  $p$ -variation, i.e.

$$(36) \quad \text{var}_p(|T'_L|) := \sup_{0 \leq x_0 < \dots < x_n \leq 1} \left( \sum_{i=0}^n \left| \frac{1}{|T'_L(x_i)|} - \frac{1}{|T'_L(x_{i-1})|} \right|^p \right)^{\frac{1}{p}} < \infty;$$

(P'2)  $\inf |T_L^{n_0'}| \geq \lambda_1 > 1$  for some  $n_0 \in \mathbb{N}$ .

We remark that, the notion of universal bounded  $p$ -variation  $\text{var}_p$  is a generalization of the usual notion of bounded variation. It is a weaker notion, allowing piecewise Holder functions. This notion is adapted to maps having  $C^{1+\alpha}$  regularity.

From these properties, it follows ([18]) that we can define a suitable strong space (the space  $S_-$  in T3.1) for the Perron-Frobenius operator  $P_T$  associated to such a  $T_L$ , in a way that it satisfies the assumptions T1, ..., T3.3 and N1. In this case, supposing a property like T3.4 then we can apply our results. For this, let us introduce a suitable space of generalized bounded variation functions with respect to the Lebesgue measure:  $BV_{1, \frac{1}{p}}$ . The functions of universal generalized bounded

variation are included in this weaker space (for more details and results see [18], in particular Lemma 2.7 for a comparison of the two spaces). A piecewise expanding map satisfying assumptions (P'1) and (P'2) has an invariant measure with density in this weaker space, moreover the transfer operator restricted to this space satisfies a Lasota-Yorke inequality and other interesting properties, as we will see in the following.

**7.1. Definition.** For an arbitrary function  $h : I \longrightarrow \mathbb{C}$  and  $\epsilon > 0$  define  $\text{osc}(h, B_\epsilon(x)) : I \longrightarrow [0, \infty]$  by

$$(37) \quad \text{osc}(h, B_\epsilon(x)) = \text{ess sup}\{|h(y_1) - h(y_2)|; y_1, y_2 \in B_\epsilon(x)\},$$

where  $B_\epsilon(x)$  denotes the open ball of center  $x$  and radius  $\epsilon$  and the essential supremum is taken with respect to the product measure  $m^2$  on  $I^2$ . Also define the real function  $\text{osc}_1(h, \epsilon)$ , on the variable  $\epsilon$ , by

$$\text{osc}_1(h, \epsilon) = \int \text{osc}(h, B_\epsilon(x)) dm(x).$$

**7.2. Definition.** Fix  $A_1 > 0$  and denote by  $\Phi$  the class of all isotonic maps  $\phi : (0, A_1] \longrightarrow [0, \infty]$ , i.e. such that  $x \leq y \implies \phi(x) \leq \phi(y)$  and  $\phi(x) \longrightarrow 0$  if  $x \longrightarrow 0$ . Set

- $R_1 = \{h : I \longrightarrow \mathbb{C}; \text{osc}_1(h, \cdot) \in \Phi\}$ ;
- For  $n \in \mathbb{N}$ , define  $R_{1,n,p} = \{h \in R_1; \text{osc}_1(h, \epsilon) \leq n \cdot \epsilon^{\frac{1}{p}} \quad \forall \epsilon \in (0, A_1]\}$ ;
- And set  $S_{1,p} = \bigcup_{n \in \mathbb{N}} R_{1,n,p}$ .

**7.3. Definition.** Let us consider the following spaces and semi-norms:

- (1)  $BV_{1,\frac{1}{p}}$  is the space of  $m$ -equivalence classes of functions in  $S_{1,p}$ ;
- (2) Let  $h : I \longrightarrow \mathbb{C}$  be a Borel function. Set

$$(38) \quad \text{var}_{1,\frac{1}{p}}(h) = \sup_{0 \leq \epsilon \leq A_1} \left( \frac{1}{\epsilon^{\frac{1}{p}}} \text{osc}_1(h, \epsilon) \right).$$

Let us consider  $|\cdot|_{1,\frac{1}{p}} : BV_{1,\frac{1}{p}} \longrightarrow \mathbb{R}$  defined by

$$(39) \quad |f|_{1,\frac{1}{p}} = \text{var}_{1,\frac{1}{p}}(f) + |f|_1,$$

it holds the following

**7.4. Proposition.**  $(BV_{1,\frac{1}{p}}, |\cdot|_{1,\frac{1}{p}})$  is a Banach space.

In the above setting, G. Keller has shown (see [18]) that there is an  $A_1 > 0$  (we recall that definition 7.2 depends on  $A_1$ ) such that:

- (a)  $BV_{1,\frac{1}{p}} \subset L^1$  is  $P_T$ -invariant,  $P_T : BV_{1,\frac{1}{p}} \longrightarrow BV_{1,\frac{1}{p}}$  is continuous and it holds  $|\cdot|_1 \leq |\cdot|_{1,\frac{1}{p}}$ ;
- (b) The unit ball of  $(BV_{1,\frac{1}{p}}, |\cdot|_{1,\frac{1}{p}})$  is relatively compact in  $(L^1, |\cdot|_1)$ ;
- (c) There exists  $k \in \mathbb{N}$ ,  $0 < \beta_0 < 1$  and  $C > 0$  such that

$$(40) \quad |P_T^k f|_{1,\frac{1}{p}} \leq \beta_0 |f|_{1,\frac{1}{p}} + C |f|_1.$$

(c)

Repeating the proof of inequality (7), we get

$$(41) \quad |P_T^n f|_{1, \frac{1}{p}} \leq B_3 \beta_2^n |f|_{1, \frac{1}{p}} + C_2 |f|_1, \quad \forall n, \quad \forall f \in BV_{1, \frac{1}{p}},$$

for  $B_3, C_2 > 0$  and  $0 < \beta_2 < 1$ .

Moreover, in [2] (Lemma 2), it was shown that

$$(d) \quad |\cdot|_\infty \leq A_1^{\frac{1}{p}-1} |\cdot|_{1, \frac{1}{p}}.$$

By this, it follows that the properties  $T1, T2, T3.1, \dots, T3.3, N1$  of section 2 are satisfied with  $S_- = BV_{1, \frac{1}{p}}$  and we can apply our construction to such maps.

We hence set

$$(42) \quad \mathcal{BV}_{1, \frac{1}{p}} := \left\{ \mu \in \mathcal{L}^1; \text{var}_{1, \frac{1}{p}}(\pi_x(\mu)) < \infty \right\}$$

and consider  $\|\cdot\|_{1, \frac{1}{p}} : \mathcal{BV}_{1, \frac{1}{p}} \rightarrow \mathbb{R}$ , defined by

$$(43) \quad \|\mu\|_{1, \frac{1}{p}} = |\phi_x|_{1, \frac{1}{p}} + \|\mu\|_1.$$

Clearly,  $(\mathcal{BV}_{1, \frac{1}{p}}, \|\cdot\|_{1, \frac{1}{p}})$  is a normed space. If we suppose that the system,  $T_L : I \rightarrow I$ , satisfies  $T3.4$ , then the system has an unique invariant probability measure with density  $\varphi_x \in BV_{1, \frac{1}{p}}$ .

Directly from the above settings, Proposition 5.8 and from Theorem 6.1 it follows convergence to equilibrium and spectral gap for these kind of maps.

**7.5. Proposition** (Exponential convergence to equilibrium). *If  $F_L$  satisfies assumptions  $G1, T1, T2, T3.4, P'1$  and  $P'2$ , then there exist  $D_2 > 0$  and  $0 < \beta_2 < 1$  such that, for every signed measure  $\mu \in \mathcal{V}$ , it holds*

$$\|F_L^{*n} \mu\|_1 \leq D_2 \beta_1^n \|\mu\|_{1, \frac{1}{p}}$$

for all  $n \geq 1$ .

**7.6. Theorem** (Spectral gap for  $\mathcal{BV}_{1, \frac{1}{p}}$ ). *If  $F_L$  satisfies assumptions  $G1, T1, T2, T3.4, P'1$  and  $P'2$ , then the operator  $F_L^* : \mathcal{BV}_{1, \frac{1}{p}} \rightarrow \mathcal{BV}_{1, \frac{1}{p}}$  can be written as*

$$F_L^* = P + N$$

where

- a)  $P$  is a projection i.e.  $P^2 = P$  and  $\dim \text{Im}(P) = 1$ ;
- b) there are  $0 < \xi < 1$  and  $K > 0$  such that for all  $\mu \in \mathcal{BV}_{1, \frac{1}{p}}$

$$\|N^n(\mu)\|_{\mathcal{BV}_{1, \frac{1}{p}}} \leq \xi^n K \|\mu\|_{\mathcal{BV}_{1, \frac{1}{p}}};$$

- c)  $P N = N P = 0$ .

We can get the same kind of results for stronger  $L^\infty$  like norms. Let us consider

$$(44) \quad \mathcal{BV}_{1, \frac{1}{p}}^\infty := \left\{ \mu \in \mathcal{L}^\infty; \frac{d(\pi_x^* \mu)}{dm} \in BV_{1, \frac{1}{p}} \right\}$$

and the function,  $\|\cdot\|_{1, \frac{1}{p}}^\infty : \mathcal{BV}_{1, \frac{1}{p}}^\infty \rightarrow \mathbb{R}$ , defined by

$$(45) \quad \|\mu\|_{1, \frac{1}{p}}^\infty = |\phi_x|_{1, \frac{1}{p}} + \|\mu\|_\infty.$$

Applying Theorem 6.2 we get

**7.7. Theorem** (Spectral gap for  $\mathcal{BV}_{1,\frac{1}{p}}^\infty$ ). *If  $F_L$  satisfies the assumptions  $G1, T1, T2, T3.4, P'1$  and  $P'2$ , then the operator  $F_L^* : \mathcal{BV}_{1,\frac{1}{p}}^\infty \rightarrow \mathcal{BV}_{1,\frac{1}{p}}^\infty$  can be written as*

$$F_L^* = P + N,$$

where

- a)  $P$  is a projection i.e.  $P^2 = P$  and  $\dim \text{Im}(P) = 1$ ;
- b) there are  $0 < \xi_1 < 1$  and  $K_2 > 0$  such that for all  $\mu \in \mathcal{BV}_{1,\frac{1}{p}}^\infty$

$$\|N^n(\mu)\|_{1,\frac{1}{p}}^\infty \leq \xi_1^n K_2 \|\mu\|_{1,\frac{1}{p}}^\infty;$$

- c)  $PN = NP = 0$ .

## 8. QUANTITATIVE STATISTICAL STABILITY

Through this section, we consider small perturbations of the transfer operator of a given system and try to study the dependence of the physical invariant measure with respect to the perturbation. A classical tool that can be applied for this type of problems is the Keller-Liverani stability theorem [19]. Since in our setting the strong space is not compactly immersed in the weak space, we cannot directly apply it. We will use another approach giving us precise bounds on the statistical stability. In this section, this approach will be applied to a class of Lorenz-like maps with slightly stronger regularity assumptions than used in Section 7.

The following is a general *quantitative* result relating the *stability* of the invariant measure for a uniform family of operators and the *convergence to equilibrium*. Let  $L$  be a transfer operator acting on two vector subspaces of signed measures on  $X$ ,  $L : (B_s, \|\cdot\|_s) \rightarrow (B_s, \|\cdot\|_s)$  and  $L : (B_w, \|\cdot\|_w) \rightarrow (B_w, \|\cdot\|_w)$  endowed with two norms, the strong norm  $\|\cdot\|_s$  on  $B_s$ , and the weak norm  $\|\cdot\|_w$  on  $B_w$ , such that  $\|\cdot\|_s \geq \|\cdot\|_w$ . Suppose that

$$B_s \subseteq B_w \subseteq \mathcal{SB}(X),$$

where  $\mathcal{SB}(X)$  denotes the space of signed Borel measures on  $X$ .

**8.1. Definition.** A one parameter family of operators  $\{L_\delta\}_{\delta \in [0,1]}$  is said to be a **uniform family of operators** if

**UF1** Let  $f_\delta \in B_s$  be a fixed probability measure for the operator  $L_\delta$ . Suppose there is  $M > 0$  such that for all  $\delta \in [0,1]$ , it holds

$$\|f_\delta\|_s \leq M;$$

**UF2**  $L_\delta$  approximates  $L_0$  when  $\delta$  is small in the following sense: there is  $C \in \mathbb{R}^+$  such that:

$$(46) \quad \|(L_0 - L_\delta)f_\delta\|_w \leq \delta C;$$

**UF3**  $L_0$  has exponential convergence to equilibrium with respect to the norms  $\|\cdot\|_s$  and  $\|\cdot\|_w$ : there exists  $0 < \rho_2 < 1$  and  $C_2 > 0$  such that for all  $f \in \mathcal{V}_s$  it holds

$$\|L_0^n f\|_w \leq \rho_2^n C_2 \|f\|_s;$$

**UF4** The iterates of the operators are uniformly bounded for the weak norm: there exists  $M_2 > 0$  such that

$$\forall \delta, n, g \in B_s \text{ it holds } \|L_\delta^n g\|_w \leq M_2 \|g\|_w.$$

We will see, under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when  $L_0$  is perturbed to  $L_\delta$ , for small values of  $\delta$ . Moreover, the modulus of continuity can be estimated.

Let us state a general lemma on the stability of fixed points satisfying certain assumptions. Let us consider two operators  $L_0$  and  $L_\delta$  preserving a normed space of signed measures  $\mathcal{B} \subseteq \mathcal{SB}(X)$  with norm  $\|\cdot\|_{\mathcal{B}}$ . Suppose that  $f_0, f_\delta \in \mathcal{B}$  are fixed points, respectively of  $L_0$  and  $L_\delta$ .

**8.2. Lemma.** *Suppose that:*

- a)  $\|L_\delta f_\delta - L_0 f_\delta\|_{\mathcal{B}} < \infty$ ;
- b)  $L_0^i$  is continuous on  $\mathcal{B}$ ;  $\exists C_i$  s.t.  $\forall g \in \mathcal{B}, \|L_0^i g\|_{\mathcal{B}} \leq C_i \|g\|_{\mathcal{B}}$ .

*Then, for each  $N$*

$$(47) \quad \|f_\delta - f_0\|_{\mathcal{B}} \leq \|L_0^N(f_\delta - f_0)\|_{\mathcal{B}} + \|L_\delta f_\delta - L_0 f_\delta\|_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i.$$

*Proof.* The proof is a direct computation

$$\begin{aligned} \|f_\delta - f_0\|_{\mathcal{B}} &\leq \|L_\delta^N f_\delta - L_0^N f_0\|_{\mathcal{B}} \\ &\leq \|L_0^N f_0 - L_0^N f_\delta\|_{\mathcal{B}} + \|L_0^N f_\delta - L_\delta^N f_\delta\|_{\mathcal{B}} \\ &\leq \|L_0^N(f_0 - f_\delta)\|_{\mathcal{B}} + \|L_0^N f_\delta - L_\delta^N f_\delta\|_{\mathcal{B}} \end{aligned}$$

(applying item b)). Hence,

$$\|f_0 - f_\delta\|_{\mathcal{B}} \leq \|L_0^N(f_0 - f_\delta)\|_{\mathcal{B}} + \|L_0^N f_\delta - L_\delta^N f_\delta\|_{\mathcal{B}}$$

but

$$L_0^N - L_\delta^N = \sum_{k=1}^N L_0^{(N-k)} (L_0 - L_\delta) L_\delta^{(k-1)}$$

hence

$$\begin{aligned} (L_0^N - L_\delta^N)f &= \sum_{k=1}^N L_0^{(N-k)} (L_0 - L_\delta) L_\delta^{*(k-1)} f_\delta \\ &= \sum_{k=1}^N L_0^{(N-k)} (L_0 - L_\delta) f_\delta \end{aligned}$$

by item c), hence

$$\begin{aligned} \|(L_0^N - L_\delta^N)f_\delta\|_{\mathcal{B}} &\leq \sum_{k=1}^N C_{N-k} \|(L_0 - L_\delta)f_\delta\|_{\mathcal{B}} \\ &\leq \|(L_0 - L_\delta)f_\delta\|_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i \end{aligned}$$

by item a), and then

$$\|f_\delta - f_0\|_{\mathcal{B}} \leq \|L_0^N(f_0 - f_\delta)\|_{\mathcal{B}} + \|(L_0 - L_\delta)f_\delta\|_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i.$$

□

Now, let us apply the statement to our family of operators satisfying assumptions UF 1,...,4, supposing  $\mathcal{B}_w = \mathcal{B}$ . We have the following

**8.3. Proposition.** *Suppose  $\{L_\delta\}_{\delta \in [0,1]}$  is a uniform family of operators as in Definition 8.1, where  $f_0$  is the unique invariant measure of  $L_0$  in  $B_w$  and  $f_\delta$  is an invariant measure of  $L_\delta$ . Then*

$$\|f_\delta - f_0\|_w = O(\delta \log \delta).$$

*Proof.* Let us apply Lemma 8.2. By UF2,

$$\|L_\delta f_\delta - L_0 f_\delta\|_w \leq \delta C$$

(see Lemma 8.2, item a) ). Moreover by UF4,  $C_i \leq M_2$ .

Hence,

$$\|f_\delta - f_0\|_w \leq \delta C M_2 N + \|L_0^N(f_0 - f_\delta)\|_w.$$

By the exponential convergence to equilibrium of  $L_0$  (UF3), there exists  $0 < \rho_2 < 1$  and  $C_2 > 0$  such that (recalling that by UF1  $\|(f_\delta - f_0)\|_s \leq 2M$ )

$$\begin{aligned} \|L_0^N(f_\delta - f_0)\|_w &\leq C_2 \rho_2^N \|(f_\delta - f_0)\|_s \\ &\leq 2C_2 \rho_2^N M \end{aligned}$$

hence

$$\|f_\delta - f_0\|_B \leq \delta C M_2 N + 2C_2 \rho_2^N M$$

choosing  $N = \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor$

$$\begin{aligned} (48) \quad \|f_\delta - f_0\|_B &\leq \delta C M_2 \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor + 2C_2 \rho_2^{\left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor} M \\ &\leq \delta \log \delta C M_2 \frac{1}{\log \rho_2} + 2C_2 \delta M. \end{aligned}$$

□

**8.1. Quantitative stability of Lorenz-like maps.** Here we apply the general results on uniform families of operators to a suitable family of bounded variation Lorenz-like maps. We consider maps as defined in Section 7, with some further assumptions.

**8.4. Definition.** A map  $F_L : [0,1]^2 \rightarrow [0,1]^2$ ,  $F_L(x, y) = (T_L(x), G_L(x, y))$ , is said to be a **BV Lorenz-like map** if it satisfies

- (1) There are  $H \geq 0$  and a partition  $\mathcal{P}' = \{J_i := (b_{i-1}, b_i), i = 1, \dots, d\}$  of  $I$  such that for all  $x_1, x_2 \in J_i$  and for all  $y \in I$  the following inequality holds

$$(49) \quad |G_L(x_1, y) - G_L(x_2, y)| \leq H \cdot |x_1 - x_2|;$$

- (2)  $F_L$  satisfy property G1 (hence is uniformly contracting on each leaf  $\gamma$  with rate of contraction  $\alpha$ );
- (3)  $T_L : I \rightarrow I$  is a piecewise expanding map satisfying the assumptions given in the following definition 8.5.

The following definition characterizes a class of piecewise expanding maps of the interval with bounded variation derivative  $T_L : I \rightarrow I$  which is a subclass of the ones considered in section 7.0.1.

**8.5. Definition** (Piecewise expanding functions with bounded variation inverse of the derivative). Suppose there exists a partition  $\mathcal{P} = \{P_i := (a_{i-1}, a_i), i = 1, \dots, q\}$  of  $I$  s.t.  $T_L : I \rightarrow I$  satisfies the following conditions. For all  $i$

- 1)  $T_{L_i} = T_L|_{P_i}$  is of class  $C^1$  and  $g_i = \frac{1}{|T_{L_i}'|}$  satisfies (P'1) of section 7, for  $p = 1$ .
- 2)  $T_L$  satisfies (P'2) of section 7:  $\inf |T_L^{n_0'}| \geq \lambda_1 > 1$  for some  $n_0 \in \mathbb{N}$ .
- 3)  $T_L$  satisfies T3.4;

In particular we assume that  $T_{L_i}$  and  $g_i$  admit a continuous extension to  $\overline{P_i} = [a_{i-1}, a_i]$  for all  $i = 1, \dots, q$ .

**8.6. Remark.** The definition 8.5 allows infinite derivative for  $T_L$  at the extreme points of its regularity intervals.

If  $F_L$  is a BV Lorenz-like map, then it satisfies G1, T3.4, P'1 and P'2. Thus, we apply the results obtained in section 7 (for  $p=1$ ) considering  $(BV_{1,1}, |\cdot|_{1,1})$  as the basis space  $(S_-, |\cdot|_s)$ , where  $|\psi|_{1,1} = \text{var}_{1,1}(\psi) + |\psi|_1$ ,  $(\mathcal{BV}_{1,1}, \|\cdot\|_{1,1})$  as the strong space  $S^1$  where  $\|\mu\|_{1,1} = |\psi|_{1,1} + \|\mu\|_1$  and finally  $(\mathcal{BV}_{1,1}^\infty, \|\cdot\|_{1,1}^\infty)$  as the strong space  $(S^\infty, \|\cdot\|_{S^\infty})$ , where  $\|\mu\|_{1,1}^\infty = |\psi|_{1,1} + \|\mu\|_\infty$ . Under these settings we have all results of section 7.

Let us define the Skorokhod distance between two piecewise expanding maps  $T_1$  and  $T_2$ . Set

$$\mathcal{C}(T_1, T_2) = \left\{ \epsilon : \exists A_1 \subseteq I \text{ and } \exists \sigma : I \rightarrow I \text{ (a diffeomorphism) s.t. } m(A_1) \geq 1 - \epsilon, \right. \\ \left. T_1|_{A_1} = T_2 \circ \sigma|_{A_1} \text{ and } \forall x \in A_1, |\sigma(x) - x| \leq \epsilon, \left| \frac{1}{\sigma'(x)} - 1 \right| \leq \epsilon \right\}$$

and

$$(50) \quad d_S(T_1, T_2) = \inf \{ \epsilon | \epsilon \in \mathcal{C}(T_1, T_2) \}.$$

By [28], Lemma 11.2.1, it follows that:

$$|P_{T_0} - P_{T_\delta}|_{BV \rightarrow L^1} \leq 14d_S(T_1, T_2).$$

Now we consider a uniform family of maps satisfying uniform Lasota-Yorke inequalities on the basis and other requirements.

**8.7. Definition.** A family of maps  $\{F_\delta\}_{\delta \in [0,1]}$  is said to be a **Uniform BV Lorenz-like family** if  $F_\delta$  is a BV Lorenz-like map (see definition 8.4) for all  $\delta \in [0,1]$  and  $\{F_\delta\}_\delta$  satisfies the following assumptions:

- (UBV1): there exist  $0 < \lambda < 1$  and  $D > 0$  s.t. for all  $f \in BV_{1,1}$  and for all  $\delta \in [0,1]$  it holds  $|P_{T_\delta}^n f|_{1,1} \leq D\lambda^n |f|_{1,1} + D|f|_1$  for all  $n \geq 1$ , where  $P_{T_\delta}$  is the Perron-Frobenius operators of  $T_\delta$ ;

When  $\delta$  is small

- (UBV2):  $d_S(T_0, T_\delta) \leq \delta$  (assumptions on  $T_\delta$ )  
 (UBV3): there is a set  $A_2$  such that  $m(A_2) \geq 1 - \delta$  and for all  $x \in A_2, y \in I$  :  $|G_0(x, y) - G_\delta(x, y)| \leq \delta$ .

For all  $\delta \in [0,1]$ , let  $n_0 = n_0(\delta) \in \mathbb{N}$  and  $\lambda_1(\delta)$  be such that  $|T_{\delta,i}^{n_0'}(x)| \geq \lambda_1(\delta) > 1$  for all  $x \in P_i$  and for each  $i = 1, \dots, q$ , where  $T_{\delta,i}^{n_0} := T_\delta^{n_0}|_{P_i}$ . Also set  $g_{i,\delta} = \frac{1}{|T_{\delta,i}'|}$  and denote by  $H_\delta > 0$  the ‘‘Lipschitz’’ constant (see item (1) of definition 8.4) of  $G_\delta$ .

- (UBV4): Suppose that:

$$(1) \quad \inf_\delta \lambda_1(\delta) > 1;$$

- (2) there exists  $C_4 > 0$  such that  $\text{var } g_{\delta,i} \leq C_4$  for all  $i = 1, \dots, q$  and all  $\delta \in [0, 1)$ ;
- (3)  $\sup_{\delta \in [0,1)} \{n_0(\delta)\} < \infty$ <sup>9</sup>
- (4)  $H_\delta \leq \overline{H}_2$  for all  $\delta \in [0, 1)$ .

**8.8. Remark.** By (UBV1) there exist  $0 < \alpha_1 < 1$  and  $\overline{D} > 0$  s.t. for all  $\mu \in \mathcal{BV}_{1,1}$  and for all  $\delta$  it holds  $\|F_\delta^{*n} \mu\|_{1,1} \leq \overline{D} \alpha_1^n \|\mu\|_{1,1} + \overline{D} \|\mu\|_1$ , for all  $n \geq 1$ . Indeed, by Lemma 5.1 we have

$$\begin{aligned} \|F_\delta^{*n} \mu\|_{1,1} &= |P_{T_\delta}^n \phi_x|_{1,1} + \|F_\delta^{*n} \mu\|_1 \\ &\leq D \lambda^n |\phi_x|_{1,1} + D |\phi_x|_1 + \|\mu\|_1 \\ &\leq D \lambda^n \|\mu\|_{1,1} + (D + 1) \|\mu\|_1. \end{aligned}$$

Now, we see that the transfer operators related to such uniform families of maps satisfy UF1,...UF4 and then allows us to apply Proposition 8.3, choosing  $\|\cdot\|_1 = \|\cdot\|_w$  as a weak norm and  $\|\cdot\|_{1,1}$  as the strong norm  $\|\cdot\|_s$ . We remark that:

- (1) UF1 easily follows by a uniform Lasota-Yorke inequality (remark 8.8);
- (2) UF3 depends only on the first element  $L_0$  of the family, and it is proved in Theorem 7.6 (see also Proposition 5.8) for transfer operators associated to Lorenz-like maps;
- (3) UF4 depends on the weak norm, and an estimation is provided in Proposition 5.1.

Some work is necessary for the property UF2. In the Proposition 8.19 we see that this property is indeed satisfied. Before state that Proposition, we need some additional concepts and results. For this, we introduce a space of measures having bounded variation in some stronger sense, and prove that the invariant measure of a BV Lorenz-like map is in it.

We have seen that a positive measure on the square,  $[0, 1]^2$ , can be disintegrated along the stable leaves  $\mathcal{F}^s$  in a way that we can see it as a family of positive measures on the interval,  $\{\mu|_\gamma\}_{\gamma \in \mathcal{F}^s}$ . Since there is a one-to-one correspondence between  $\mathcal{F}^s$  and  $[0, 1]$ , this defines a path in the space of positive measures,  $[0, 1] \mapsto \mathcal{SB}(I)$ . It will be convenient to use a functional notation and denote such a path by  $\Gamma_\mu : I \rightarrow \mathcal{SB}(I)$  defined almost everywhere by  $\Gamma_\mu(\gamma) = \mu|_\gamma$ , where  $(\{\mu_\gamma\}_{\gamma \in I}, \phi_x)$  is some disintegration for  $\mu$ . However, since such a disintegration is defined  $\mu_x$ -a.e.  $\gamma \in [0, 1]$ , the path  $\Gamma_\mu$  is not unique. For this reason we define more precisely  $\Gamma_\mu$  as the class of almost everywhere equivalent paths corresponding to  $\mu$ .

**8.9. Definition.** Consider a Borel measure  $\mu$  and a disintegration  $\omega = (\{\mu_\gamma\}_{\gamma \in I}, \phi_x)$ , where  $\{\mu_\gamma\}_{\gamma \in I}$  is a family of probabilities on  $\Sigma$  defined  $\mu_x$ -a.e.  $\gamma \in I$  (where  $\mu_x = \phi_x m$ ) and  $\phi_x : I \rightarrow \mathbb{R}$  is a non-negative marginal density, as before. Denote by  $\Gamma_\mu$  the class of equivalent paths associated to  $\mu$

$$\Gamma_\mu(\gamma) = \{\Gamma_\mu^\omega\},$$

where  $\omega$  ranges on all the possible disintegrations of  $\mu$  on the stable foliation and  $\Gamma_\mu^\omega : I \rightarrow \mathcal{SB}(I)$  is the path associated to a given disintegration:

$$\Gamma_\mu^\omega(\gamma) = \mu|_\gamma = \pi_{\gamma,y}^* \phi_x(\gamma) \mu_\gamma.$$

<sup>9</sup>For instance, it holds if  $\inf |T'_{\delta,i}(x)| > 1$  for all  $\delta \in [0, 1)$  and for all  $i = 1, \dots, q$ .



In the following, when no ambiguity is possible we will consider informally  $\Gamma_\mu$  itself as a path. Let us call the set on which  $\Gamma_\mu^\omega$  is defined by  $I_{\Gamma_\mu^\omega}$ .

**8.10. Definition.** Let  $\mathcal{P} = \mathcal{P}(\Gamma_\mu^\omega)$  be a finite sequence  $\mathcal{P} = \{x_i\}_{i=1}^n \subset I_{\Gamma_\mu^\omega}$  and define the **variation of  $\Gamma_\mu^\omega$  with respect to  $\mathcal{P}$**  as (denote  $\gamma_i := \gamma_{x_i}$ )

$$\text{Var}(\Gamma_\mu^\omega, \mathcal{P}) = \sum_{j=1}^n \|\Gamma_\mu^\omega(\gamma_j) - \Gamma_\mu^\omega(\gamma_{j-1})\|_W,$$

where we recall  $\|\cdot\|_W$  is the Wasserstein-like norm defined by equation (9). Finally we define the **variation of  $\Gamma_\mu^\omega$**  by taking the supremum over the sequences, as

$$\text{Var}(\Gamma_\mu^\omega) := \sup_{\mathcal{P}} \text{Var}(\Gamma_\mu^\omega, \mathcal{P}).$$

**8.11. Remark.** For an interval  $\eta \subset I$ , we define

$$\text{Var}_{\bar{\eta}}(\Gamma_\mu^\omega) := \text{Var}(\Gamma_\mu^\omega|_{\bar{\eta}}),$$

where  $\bar{\eta}$  is the closure of  $\eta$ .

**8.12. Remark.** When no confusion can be done, to simplify the notation, we denote  $\Gamma_\mu^\omega(\gamma)$  just by  $\mu|_\gamma$ .

**8.13. Definition.** Define the **variation of a positive measure  $\mu$**  by

$$\text{Var}(\mu) = \inf_{\Gamma_\mu^\omega \in \Gamma_\mu} \{\text{Var}(\Gamma_\mu^\omega)\}.$$

We remark that,

$$\|\mu\|_1 = \int W_0^1(0, \Gamma_\mu^\omega(\gamma)) dm(\gamma), \quad \text{for any } \Gamma_\mu^\omega \in \Gamma_\mu.$$

**8.14. Definition.** From the definition 8.10 we define the set of bounded variation positive measures  $\mathcal{BV}^+$  as

$$(51) \quad \mathcal{BV}^+ = \{\mu \in \mathcal{AB} : \mu \geq 0, \text{Var}(\mu) < \infty\}.$$

Now we are ready to state a lemma estimating the regularity of the iterates  $F^{*n}(m)$ . Next result is a Lasota-Yorke-like inequality where the strong semi-norm is the variation  $\text{Var}(\mu)$  defined in 8.13. This is our main tool to estimate the regularity of the invariant measure of a BV Lorenz-like map. The proof will be postponed to the appendix (see Proposition 9.13).

**8.15. Proposition.** *Let  $F_L(x, y) = (T_L(x), G_L(x, y))$  be a BV Lorenz-like map. Then, there are  $C_0, 0 < \lambda_0 < 1$  and  $k \in \mathbb{N}$  such that for all  $\mu \in \mathcal{BV}^+$  and all  $n \geq 1$  it holds (denote  $\tilde{F} := F^k$ )*

$$(52) \quad \text{Var}(\tilde{F}^{*n} \mu) \leq C_0 \lambda_0^n \text{Var}(\mu) + C_0 |\phi_x|_{1,1}.$$

A precise estimate for  $C_0$  can be found in equation (69). Remember, by Proposition 5.10, a Lorenz-like map has an invariant measure  $\mu_0 \in S^\infty$ .

**8.16. Proposition.** *Let  $F_L(x, y) = (T_L(x), G_L(x, y))$  be BV Lorenz-like map and suppose that  $F_L$  has an unique invariant probability measure  $\mu_0 \in \mathcal{BV}_{1,1}^\infty$ . Then  $\mu_0 \in \mathcal{BV}^+$  and*

$$\text{Var}(\mu_0) \leq 2C_0.$$

*Proof.* Let  $\tilde{F} := F_L^k$  where  $k$  comes from Proposition 8.15. Then  $\mu_0 \in \mathcal{BV}_{1,1}^\infty$  is the unique  $\tilde{F}$ -invariant probability measure in  $\mathcal{BV}_{1,1}^\infty$ . Consider the Lebesgue measure  $m$  and the iterates  $\tilde{F}^{*n}(m)$ . By Theorem 7.7, these iterates converge to  $\mu_0$  in  $\mathcal{L}^\infty$ . It means that the sequence  $\{\Gamma_{\tilde{F}^{*n}(m)}^\omega\}_n$  converges  $m$ -a.e. to  $\Gamma_{\mu_0}^\omega \in \Gamma_{\mu_0}$ , where  $\Gamma_{\mu_0}^\omega$  is a path given by the Rokhlin Disintegration Theorem and  $\{\Gamma_{\tilde{F}^{*n}(m)}^\omega\}_n$  is given by remark 4.2. It implies that  $\{\Gamma_{\tilde{F}^{*n}(m)}^\omega\}_n$  converges pointwise to  $\Gamma_{\mu_0}^\omega$  on a full measure set  $\hat{I} \subset I$ . Let us denote  $\widehat{\Gamma}_n^\omega = \Gamma_{\tilde{F}^{*n}(m)}^\omega|_{\hat{I}}$  and  $\widehat{\Gamma}_{\mu_0}^\omega = \Gamma_{\mu_0}^\omega|_{\hat{I}}$ . Since  $\{\widehat{\Gamma}_n^\omega\}_n$  converges pointwise to  $\widehat{\Gamma}_{\mu_0}^\omega$  it holds  $\text{Var}(\widehat{\Gamma}_n^\omega, \mathcal{P}) \rightarrow \text{Var}(\widehat{\Gamma}_{\mu_0}^\omega, \mathcal{P})$  as  $n \rightarrow \infty$  for all partition  $\mathcal{P}$ . On the other hand  $\text{Var}(\widehat{\Gamma}_n^\omega, \mathcal{P}) \leq \text{Var}(\tilde{F}^{*n}(m)) \leq 2C_0$  for all  $n \geq 1$ , where  $C_0$  comes from Proposition 8.15. Then  $\text{Var}(\widehat{\Gamma}_{\mu_0}^\omega, \mathcal{P}) \leq 2C_0$  for all partition  $\mathcal{P}$ . Thus  $\text{Var}(\widehat{\Gamma}_{\mu_0}^\omega) \leq 2C_0$  and hence  $\text{Var}(\mu_0) \leq 2C_0$ .  $\square$

**8.17. Remark.** We remark that, Proposition 8.16 is an estimation of the regularity of the disintegration of  $\mu_0$ . Similar results are presented in [16] and [11].

The proof of the following proposition is postponed to the appendix (see Proposition 9.14).

**8.18. Proposition.** *Let  $\{F_\delta\}_\delta$ ,  $F_\delta = (T_\delta, G_\delta)$  be a Uniform BV Lorenz-like family (definition (8.7)) and let  $f_\delta$  be the unique  $F_\delta$ -invariant probability in  $\mathcal{BV}_{1,1}^\infty$ . Then, there exists  $\overline{C}_0 > 0$  such that*

$$(53) \quad \text{Var}(f_\delta) \leq \overline{C}_0,$$

for all  $\delta \in [0, 1)$ .

We are now ready to prove the following

**8.19. Proposition** (to obtain UF2). *Let  $\{F_\delta\}_\delta$ ,  $F_\delta = (T_\delta, G_\delta)$ , be a family of BV Lorenz-like maps which satisfy UBV2 and UBV3 of definition 8.7. Denote by  $F_\delta^*$  their transfer operators and by  $f_\delta$  their fixed points in  $\mathcal{BV}_{1,1}^\infty$ . Suppose that  $f_\delta$  has uniformly bounded variation,*

$$\text{Var}(f_\delta) \leq M_2, \quad \forall \delta.$$

Then, there is a constant  $C_1$  such that for  $\delta$  small enough

$$\|(F_0^* - F_\delta^*)f_\delta\|_1 \leq C_1\delta(M_2 + 1).$$

*Proof.* Set  $A = A_1 \cap A_2$ , where  $A_1$  comes from the definition of  $\mathcal{C}(T_1, T_2)$  (see equation (50)) and  $A_2$  is from (UBV3) (see definition 8.7). Let us estimate

$$(54) \quad \|(F_0^* - F_\delta^*)f_\delta\|_1 = \int_I \|F_0^*(1_A f_\delta)|_\gamma - F_\delta^*(1_A f_\delta)|_\gamma\|_W dm(\gamma) + \int_I \|F_0^*(1_{A^c} f_\delta)|_\gamma - F_\delta^*(1_{A^c} f_\delta)|_\gamma\|_W dm(\gamma).$$

By the assumptions, for a.e.  $\gamma$ ,  $\|f_\delta|_\gamma\|_W \leq (M_2 + 1)$  and  $\|1_{A^c} f_\delta\|_1 \leq (M_2 + 1)\delta$ . Since  $F^*$  is a contraction for the weak norm, we have

$$\int_I \|F_0^*(1_{A^c} f_\delta)|_\gamma - F_\delta^*(1_{A^c} f_\delta)|_\gamma\|_W dm(\gamma) \leq 2(M_2 + 1)\delta.$$

Now, let us estimate the first summand of (54) by estimating the integral

$$\int ||(\mathbf{F}_0^* \mu - \mathbf{F}_\delta^* \mu)|_\gamma||_W dm(\gamma),$$

where  $\mu = 1_A f_\delta$ . Denote by  $T_{0,i}$ , with  $0 \leq i \leq q$ , the branches of  $T_0$  defined in the sets  $P_i \in \mathcal{P}$  and set  $T_{\delta,i} = T_\delta|_{P_i \cap A}$ . These functions will play the role of the branches for  $T_\delta$ . Since in  $A$ ,  $T_0 = T_\delta \circ \sigma_\delta$  (where  $\sigma_\delta$  is the diffeomorphism in the definition of the Skorokhod distance), then  $T_{\delta,i}$  are invertible. Then

$$(\mathbf{F}_0^* \mu - \mathbf{F}_\delta^* \mu)|_\gamma = \sum_{i=1}^q \frac{\mathbf{F}_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{T_0(P_i \cap A)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{T_\delta(P_i \cap A)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \quad \mu_x - a.e. \gamma \in I.$$

Let us now consider  $T_0(P_i \cap A)$ ,  $T_\delta(P_i \cap A)$  and remark that  $T_0(P_i \cap A) = \sigma_\delta(T_\delta(P_i \cap A))$  where  $\sigma_\delta$  is a diffeomorphism near to the identity. Let us denote  $B_i = T_0(P_i \cap A) \cap T_\delta(P_i \cap A)$  and  $C_i = T_0(P_i \cap A) \triangle T_\delta(P_i \cap A)$ . Then, we have

$$(55) \quad \int_I ||(\mathbf{F}_0^* \mu - \mathbf{F}_\delta^* \mu)|_\gamma||_W dm(\gamma) \leq O_1 + O_2,$$

where

$$O_1 = \int_I \left\| \sum_{i=1}^q \frac{\mathbf{F}_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm$$

and

$$O_2 = \int_I \left\| \sum_{i=1}^q \frac{\mathbf{F}_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{C_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{C_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm.$$

And since there is  $K_1$  such that  $m(C_i) \leq K_1 \delta$ , we get

$$O_2 = \int_I \left\| \sum_{i=1}^q \frac{\mathbf{F}_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{C_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{C_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm \leq q K_1 (M_2 + 1) \delta.$$

In order to estimate  $O_1$ , we note that

$$\begin{aligned} O_1 &= \int_I \left\| \sum_{i=1}^q \frac{\mathbf{F}_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm \\ &\leq \int_I \left\| \sum_{i=1}^q \frac{\mathbf{F}_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm \\ &\quad + \int_I \left\| \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathbf{F}_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm \\ &= \int_I I(\gamma) dm(\gamma) + \int_I II(\gamma) dm(\gamma). \end{aligned}$$

The two summands will be treated separately. Let us denote  $\bar{\mu}|_\gamma = \pi_{\gamma,y}^* \mu_\gamma$  (note that  $\mu|_\gamma = \phi_\mu(\gamma) \bar{\mu}|_\gamma$  and  $\bar{\mu}|_\gamma$  is a probability measure).

$$\begin{aligned}
I(\gamma) &= \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W \\
&\leq \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} \right\|_W \\
&\quad + \left\| \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right\|_W \\
&= I_a(\gamma) + I_b(\gamma).
\end{aligned}$$

Since  $f_\delta$  is a probability measure it holds, posing  $\beta = T_{0,i}^{-1}(\gamma)$

$$\begin{aligned}
\int I_a(\gamma) dm &= \int \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} \right\|_W dm(\gamma) \\
&\leq \int \sum_{i=1}^q \left\| \frac{F_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} \right\|_W dm \\
&\leq \sum_{i=1}^q \int \left\| \frac{F_{0,T_{0,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)}^* \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} \right\|_W dm \\
&\leq \sum_{i=1}^q \int_{T_{0,i}^{-1}(B_i)} \left\| F_{0,\beta}^* \mu|_\beta - F_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}^* \mu|_\beta \right\|_W dm(\beta).
\end{aligned}$$

We remark  $T_{0,i}^{-1}(B_i) \subseteq P_i \cap A$  and  $T_{\delta,i}^{-1}(T_{0,i}(T_{0,i}^{-1}(B_i))) \subseteq P_i \cap A$ . Since  $|T_{\delta,i}(\beta) - T_{0,i}(\beta)| \leq \delta$  and  $T_{0,i}^{-1}$  is a contraction, then  $|T_{0,i}^{-1} \circ T_{\delta,i}(\beta) - \beta| \leq \delta$ . Therefore

$$\begin{aligned}
\left\| F_{0,\beta}^* \mu|_\beta - F_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}^* \mu|_\beta \right\|_W &\leq \left\| F_{0,\beta}^* \mu|_\beta - F_{\delta,\beta}^* \mu|_\beta \right\|_W \\
&\quad + \left\| F_{\delta,\beta}^* \mu|_\beta - F_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}^* \mu|_\beta \right\|_W.
\end{aligned}$$

By the assumption (3),

$$\left\| F_{0,\beta}^* \mu|_\beta - F_{\delta,\beta}^* \mu|_\beta \right\|_W \leq \delta(M_2 + 1).$$

By the assumption (5)

$$\left\| F_{\delta,\beta}^* \mu|_\beta - F_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}^* \mu|_\beta \right\|_W \leq H\delta(M_2 + 1).$$

Thus,

$$I_a(\gamma) \leq (H + 1)\delta(M_2 + 1).$$

To estimate  $I_b(\gamma)$ , we have

$$\begin{aligned} I_b(\gamma) &= \left\| \sum_{i=1}^q \frac{F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{0, i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0, i}(T_{0, i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{\delta, i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} \right\|_W \\ &\leq \sum_{i=1}^q \left| \frac{\chi_{B_i}(\gamma)}{|T'_{0, i}(T_{0, i}^{-1}(\gamma))|} - \frac{\chi_{B_i}(\gamma)}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} \right| \left\| F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{0, i}^{-1}(\gamma)} \right\|_W \end{aligned}$$

and

$$\int I_b(\gamma) \, dm(\gamma) \leq |(P_{T_0} - P_{T_\delta})(1)| (M_2 + 1).$$

By [28], Lemma 11.2.1, we get

$$\int_{A_1} I_b(\gamma) \, dm(\gamma) \leq |\phi_x|_\infty |(P_{T_0} - P_{T_\delta})1|_1 \leq 14(M_2 + 1)\delta.$$

Now, let us estimate the integral of the second summand

$$II(\gamma) = \left\| \sum_{i=1}^q \frac{F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{0, i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{\delta, i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} \right\|_W.$$

Let us make the change of variable  $\gamma = T_{\delta, i}(\beta)$ .

$$\begin{aligned} \int_I II(\gamma) \, dm(\gamma) &= \int_I \left\| \sum_{i=1}^q \frac{F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{0, i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* \mu|_{T_{\delta, i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} \right\|_W dm(\gamma) \\ &\leq \sum_{i=1}^q \int_{B_i} \frac{1}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} \left\| F_{\delta, T_{\delta, i}^{-1}(\gamma)}^* (\mu|_{T_{0, i}^{-1}(\gamma)} - \mu|_{T_{\delta, i}^{-1}(\gamma)}) \right\|_W dm(\gamma) \\ &\leq \sum_{i=1}^q \int_{B_i} \frac{1}{|T'_{\delta, i}(T_{\delta, i}^{-1}(\gamma))|} \left\| \mu|_{T_{0, i}^{-1}(\gamma)} - \mu|_{T_{\delta, i}^{-1}(\gamma)} \right\|_W dm(\gamma) \\ &\leq \sum_{i=1}^q \int_{T_{\delta, i}^{-1}(B_i)} \left\| \mu|_{T_{0, i}^{-1}(T_{\delta, i}(\beta))} - \mu|_\beta \right\|_W dm(\beta). \end{aligned}$$

Since  $|T_{\delta, i}(\beta) - T_{0, i}(\beta)| \leq \delta$  and  $T_{0, i}^{-1}$  is a contraction, we have  $|T_{0, i}^{-1} \circ T_{\delta, i}(\beta) - \beta| \leq \delta$ . Hence,

$$\int_I II(\gamma) \, dm(\gamma) \leq \int \sup_{x, y \in B(\beta, \delta)} (|\mu|_x - \mu|_y|_W) dm(\beta)$$

and then

$$\int_I II(\gamma) \, dm(\gamma) \leq 2\delta(M_2 + 1).$$

Summing all, the statement is proved.  $\square$

Once this is done, we have all the ingredients to apply Proposition 8.3 and obtain the quantitative estimation.

**8.20. Theorem** (Quantitative stability for deterministic perturbations). *Let  $\{F_\delta\}_{\delta \in [0, 1]}$  be a uniform BV Lorenz-like family satisfying UB1, UB2, UB3 and UB4. Denote by  $f_\delta$  the fixed point of  $F_\delta^*$  in  $\mathcal{BV}_{1,1}^\infty$ , for all  $\delta$ . Then*

$$\|f_\delta - f_0\|_1 = O(\delta \log \delta).$$

## 9. APPENDIX 1: PROOF OF PROPOSITION 8.15

In this section, we obtain Proposition 8.15 as a particular case of Theorem 9.2. Note that, for all  $\mu \in \mathcal{BV}^+$  it holds  $\|\mu\|_1 = |\phi_x|_1$  and  $\|\mu\|_\infty = |\phi_x|_\infty$ , where  $\phi_x = \frac{d\pi_x^* \mu}{dm}$ . We also remark, for each  $\mu \in \mathcal{BV}^+$  we have  $\phi_x \in BV_{1,1}$ .

For a measurable map  $F : [0, 1]^2 \rightarrow [0, 1]^2$ , of the type  $F(x, y) = (T(x), G(x, y))$ , we denote by  $F_\gamma : [0, 1] \rightarrow [0, 1]$ , the function defined by

$$(56) \quad F_\gamma = \pi_{\gamma, y} \circ F|_\gamma \circ \pi_{\gamma, y}^{-1},$$

where  $\pi_{\gamma, y}$  is the restriction on  $\gamma$  of the projection  $\pi(x, y) = y$ .

**9.1. Definition.** Consider a function  $f : [0, 1]^2 \rightarrow \mathbb{R}$  and let  $x_1 \leq \dots \leq x_n$  and  $y_1 \leq \dots \leq y_n$  be such that  $(x_i)_{i=1}^n \subset I$  and  $(y_i)_{i=1}^n \subset I$ . We define  $\text{var}^\diamond(f, (x_i)_{i=1}^n, (y_i)_{i=1}^n)$  by

$$\text{var}^\diamond(f, (x_i)_{i=1}^n, (y_i)_{i=1}^n) := \sum_{i=1}^n |f(x_{i+1}, y_i) - f(x_i, y_i)|,$$

and

$$(57) \quad \text{var}^\diamond(f) := \sup_{(x_i)_{i=1}^n, (y_i)_{i=1}^n} \text{var}^\diamond(f, (x_i)_{i=1}^n, (y_i)_{i=1}^n).$$

If  $\eta \subset I$  is an interval, we define  $\text{var}_\eta^\diamond(f) = \text{var}^\diamond(f|_{\bar{\eta} \times I})$ , where  $\bar{\eta}$  is the closure of  $\eta$ .

Since preliminaries results are necessary, we postponed the proof of the next theorem to the end of the section.

**9.2. Theorem.** *Let  $F(x, y) = (T(x), G(x, y))$  be a measurable transformation such that*

- (1)  $\text{var}^\diamond(G) < \infty$
- (2)  $F$  satisfy property G1 (hence is uniformly contracting on each leaf  $\gamma$  with rate of contraction  $\alpha$ );
- (3)  $T : [0, 1] \rightarrow [0, 1]$  is a piecewise expanding map satisfying the assumptions given in the definition 8.5.

*Then, there are  $C_0$ ,  $0 < \lambda_0 < 1$  and  $k \in \mathbb{N}$  such that for all  $\mu \in \mathcal{BV}^+$  and all  $n \geq 1$  it holds (denote  $\tilde{F} := F^k$ )*

$$(58) \quad \text{Var}(\tilde{F}^{*n} \mu) \leq C_0 \lambda_0^n \text{Var}(\mu) + C_0 |\phi_x|_{1,1}.$$

**9.3. Remark.** If  $F_L$  is a BV Lorenz-like map (definition 8.4), a straightforward computation yields

$$\text{var}^\diamond(G_L) \leq H,$$

where  $H$  comes from equation (49). This shows that Proposition 8.15 is a direct consequence of Theorem 9.2.

**9.1. Lasota-Yorke Inequality for positive measures.** Henceforth, we fix a positive measure  $\mu \in \mathcal{BV}^+ \subset \mathcal{AB}$  and a path which represents  $\mu$  (i.e. a pair  $(\{\mu_\gamma\}_\gamma, \phi_x)$  s.t.  $\Gamma_\mu^\omega(\gamma) = \mu|_\gamma$ ). To simplify, we will denote the path  $\Gamma_\mu^\omega \in \Gamma_\mu$ , just by  $\Gamma_\mu$ .

**9.4. Remark.** Consider  $T : [0, 1] \rightarrow [1, 0]$  a piecewise expanding map from definition 8.5 and  $g_i = \frac{1}{T_i'}$ . For all  $n \geq 1$ , let  $\mathcal{P}^{(n)}$  be the partition of  $I$  s.t.  $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$  if and only if  $\mathcal{P}^{(1)}(T^j(x)) = \mathcal{P}^{(1)}(T^j(y))$  for all  $j = 0, \dots, n$ , where  $\mathcal{P}^{(1)} = \mathcal{P}$  (see definition 8.5). Given  $P \in \mathcal{P}^{(n)}$ , define  $g_P^{(n)} = \frac{1}{|T^{(n)}|_P|}$ . Item 2) implies that there exists  $C_1 > 0$  and  $0 < \theta < 1$  s.t.

$$(59) \quad \sup\{g_P^{(n)}\} \leq C_1 \theta^n, \text{ for all } P \in \mathcal{P}^{(n)} \text{ and all } n \geq 1.$$

Moreover, equation (59) and some basic properties of real valued  $BV$  functions imply (see [27], page 41, equation (3.1)) there exists  $\lambda_2 \in (\theta, 1)$  and  $C_2 > 0$  such that

$$\text{var}(g_P^{(n)}) \leq C_2 \lambda_2^n, \text{ for all } P \in \mathcal{P}^{(n)} \text{ and all } n \geq 1.$$

Henceforth, instead of  $F$  we consider an iterate of it,  $\tilde{F} := F^k$ , such that  $T^k$  satisfies

$$(60) \quad \text{var } g_P^{(k)} + 3 \sup g_P^{(k)} \leq \beta < 1, \text{ for some } 0 < \beta < 1 \ \forall P \in \mathcal{P}^{(k)}.$$

Doing so, we remark that  $G^n := \pi_y \circ F^n$  also satisfies

$$(61) \quad \text{var}^\diamond(G^n) < \infty, \ \forall n.$$

Next lemma provides equation (61) and its proof can be found in [2].

**9.5. Lemma.** *If  $F$  satisfy definition 8.4 and  $p \geq 1$ , then there are  $C, K \in \mathbb{R}$  such that*<sup>10</sup>

$$|\pi(f \circ F^n)|_{1, \frac{1}{p}} + \text{var}^\diamond(f \circ F^n) \leq CK^n(|\pi(f)|_{1, \frac{1}{p}} + |f|_{lip'} + \text{var}^\diamond(f)),$$

for each  $n \geq 1$ .

Recalling equation (56), set

$$(62) \quad \Gamma_{\mu_{\tilde{F}}}(\gamma) := \tilde{F}_\gamma^* \Gamma_\mu(\gamma).$$

With the above notation and following the strategy of the proof of Lemma 4.1, we have that the path  $\Gamma_{\tilde{F}^* \mu}$ , defined on a full measure set by

$$(63) \quad \Gamma_{\tilde{F}^* \mu}(\gamma) = \sum_{i=1}^q (g_i \cdot \Gamma_{\mu_{\tilde{F}}}) \circ T_{L_i}^{-1}(\gamma) \cdot \chi_{T_L(P_i)}(\gamma), \text{ where } g_i = \frac{1}{T_{L_i}'},$$

represents the measure  $\tilde{F}^* \mu$ .

By Lemma 5.2 and equation (56) it holds

$$\|\tilde{F}_\gamma^* \Gamma_\mu(\gamma)\|_W \leq \|\Gamma_\mu(\gamma)\|_W,$$

for  $m$ -a.e.  $\gamma \in I$ . Then we have the following

---

<sup>10</sup> $\pi(f)(x) = \int f(x, y) dm(y)$  and  $|f|_{lip'} = |f|_\infty + Lip_y(g)$ , where  $Lip_y(g) = \sup_{x, y_1, y_2 \in [0, 1]} \frac{|g(x, y_2) - g(x, y_1)|}{y_2 - y_1}$ .

**9.6. Lemma.** *Let  $\gamma_1$  and  $\gamma_2$  be two leaves such that  $\gamma_1, \gamma_2 \in P_i$  for some  $i = 1, \dots, q$  (see definition 8.4). Then for every path  $\Gamma_\mu$ , where  $\mu \in \mathcal{AB}$ , it holds*

$$(64) \quad \|\tilde{F}_{\gamma_1}^* \Gamma_\mu(\gamma_1) - \tilde{F}_{\gamma_2}^* \Gamma_\mu(\gamma_2)\|_W \leq \|\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)\|_W + |G(\gamma_1, y_0) - G(\gamma_2, y_0)| |\phi_x|_\infty,$$

for some  $y_0 \in I$ .

*Proof.* Consider  $g$  such that  $|g|_\infty \leq 1$  and  $Lip(g) \leq 1$ , and observe that since  $G_{\gamma_1} - G_{\gamma_2} : I \rightarrow I$  is continuous, it holds

$$\sup_I |G(\gamma_1, y) - G(\gamma_2, y)| = |G(\gamma_1, y_0) - G(\gamma_2, y_0)|,$$

for some  $y_0 \in I$ . Moreover, by equation (5.2) we have

$$\begin{aligned} \left| \int g d\Gamma_{\mu_{\tilde{F}}}(\gamma_1) - \int g d\Gamma_{\mu_{\tilde{F}}}(\gamma_2) \right| &= \left| \int g d\tilde{F}_{\gamma_1}^* \Gamma_\mu(\gamma_1) - \int g d\tilde{F}_{\gamma_2}^* \Gamma_\mu(\gamma_2) \right| \\ &\leq \left| \int g d\tilde{F}_{\gamma_1}^* \Gamma_\mu(\gamma_1) - \int g d\tilde{F}_{\gamma_1}^* \Gamma_\mu(\gamma_2) \right| \\ &\quad + \left| \int g d\tilde{F}_{\gamma_1}^* \Gamma_\mu(\gamma_2) - \int g d\tilde{F}_{\gamma_2}^* \Gamma_\mu(\gamma_2) \right| \\ &\leq \left\| \tilde{F}_{\gamma_1}^* (\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)) \right\|_W \\ &\quad + \int |g(\tilde{F}_{\gamma_1}) - g(\tilde{F}_{\gamma_2})| d\mu|_{\gamma_2} \\ &\leq \|\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)\|_W \\ &\quad + \int |G(\gamma_1, y) - G(\gamma_2, y)| d\mu|_{\gamma_2(y)} \\ &\leq \|\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)\|_W \\ &\quad + \sup_I |G(\gamma_1, y) - G(\gamma_2, y)| \int 1 d\mu|_{\gamma_2(y)} \\ &= \|\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)\|_W + |G(\gamma_1, y_0) - G(\gamma_2, y_0)| |\phi_x|_\infty. \end{aligned}$$

Taking the supremum over  $g$  such that  $|g|_\infty \leq 1$  and  $L(g) \leq 1$ , we finish the proof.  $\square$

The proofs of the next two lemmas are straightforward and analogous to the one dimensional BV functions. So, we omit them (details can be found in [23]).

**9.7. Lemma.** *Given paths  $\Gamma_{\mu_0}, \Gamma_{\mu_1}$  and  $\Gamma_{\mu_2}$  (where  $\Gamma_{\mu_i}(\gamma) = \mu_i|_\gamma$ ) representing the positive measures  $\mu_0, \mu_1, \mu_2 \in \mathcal{BV}^+$  respectively, a function  $\varphi : I \rightarrow \mathbb{R}$ , an homeomorphism  $h : \eta \subset I \rightarrow h(\eta) \subset I$  and a subinterval  $\eta \subset I$ , then the following properties hold*

P1) *If  $\mathcal{P}$  is a partition of  $I$  by intervals  $\eta$ , then*

$$\text{Var}(\Gamma_{\mu_0}) = \sum_{\eta} \text{Var}_{\overline{\eta}}(\Gamma_{\mu_0});$$

P2)  $\text{Var}_{\overline{\eta}}(\Gamma_{\mu_1} + \Gamma_{\mu_2}) \leq \text{Var}_{\overline{\eta}}(\Gamma_{\mu_1}) + \text{Var}_{\overline{\eta}}(\Gamma_{\mu_2})$

P3)  $\text{Var}_{\overline{\eta}}(\varphi \cdot \Gamma_{\mu_0}) \leq (\sup_{\overline{\eta}} |\varphi|) \cdot (\text{Var}_{\overline{\eta}}(\Gamma_{\mu_0})) + \left( \sup_{\gamma \in \overline{\eta}} \|\Gamma_{\mu_0}(\gamma)\|_W \right) \cdot \text{var}_{\overline{\eta}}(\varphi)$



$$\text{P4)} \quad \text{Var}_{\overline{\eta}}(\Gamma_{\mu_0} \circ h) = \text{Var}_{\overline{h(\eta)}}(\Gamma_{\mu_0}).$$

9.8. **Remark.** For every path  $\Gamma_\mu$ ,  $\mu \in \mathcal{AB}$  and an interval  $\eta \subset I$ , it holds

$$\sup_{\gamma \in \overline{\eta}} \|\Gamma_\mu(\gamma)\|_W \leq \text{Var}_{\overline{\eta}}(\Gamma_\mu) + \frac{1}{m(\overline{\eta})} \int_{\overline{\eta}} \|\Gamma_\mu(\gamma)\|_W dm(\gamma),$$

where  $\overline{\eta}$  is the closure of  $\eta$ .

9.9. **Lemma.** For all  $\Gamma_\mu$ , where  $\mu \in \mathcal{BV}^+$ , and all  $P \in \mathcal{P}$  it holds

$$\text{Var}_{\overline{P}}(\Gamma_{\mu_{\overline{P}}}) \leq \text{Var}_{\overline{P}}(\Gamma_\mu) + \text{var}_{\overline{P}}^\diamond(G) |\phi_x|_\infty.$$

*Proof.* Consider  $(\gamma_i)_{i=1}^n \subset \overline{P}$  such that  $\gamma_1 \leq \dots \leq \gamma_n$ . By Lemma 9.6, for every  $i$  there is  $y_i$  such that

$$\begin{aligned} \sum_{i=1}^n \|\tilde{F}_{\gamma_{i+1}}^* \Gamma_\mu(\gamma_{i+1}) - \tilde{F}_{\gamma_i}^* \Gamma_\mu(\gamma_i)\|_W &\leq \sum_{i=1}^n \|\Gamma_\mu(\gamma_{i+1}) - \Gamma_\mu(\gamma_i)\|_W + \sum_{i=1}^n |G(\gamma_{i+1}, y_i) - G(\gamma_i, y_i)| |\phi_x|_\infty \\ &\leq \sum_{i=1}^n \|\Gamma_\mu(\gamma_{i+1}) - \Gamma_\mu(\gamma_i)\|_W + |\phi_x|_\infty \text{var}_{\overline{\eta}}^\diamond(G). \end{aligned}$$

Then,

$$\sum_{i=1}^n \|\tilde{F}_{\gamma_{i+1}}^* \Gamma_\mu(\gamma_{i+1}) - \tilde{F}_{\gamma_i}^* \Gamma_\mu(\gamma_i)\|_W \leq \text{Var}_{\overline{P}}(\Gamma_\mu) + |\phi_x|_\infty \text{var}_{\overline{P}}^\diamond(G).$$

We finish the proof taking the supremum over  $(\gamma_i)_i^n$ . □

9.10. **Lemma.** For all path  $\Gamma_\mu$ , where  $\mu \in \mathcal{BV}^+$ , it holds

$$\text{Var}(\Gamma_{\tilde{F}^* \mu}) \leq \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \cdot \sup_{\gamma \in \overline{P_i}} \|\Gamma_\mu(\gamma)\|_W + \sup_{\overline{P_i}} g_i \cdot \text{Var}_{\overline{P_i}}(\Gamma_{\mu_{\overline{F}}}),$$

where  $\Gamma_{\mu_{\overline{F}}}$  is defined by equation (62).

*Proof.* Using the properties P2, P3,  $\sup_{\gamma \in \overline{P_i}} \|\Gamma_{\mu_{\overline{F}}}(\gamma)\|_W \leq \sup_{\gamma \in \overline{P_i}} \|\Gamma_{\mu}(\gamma)\|_W$  and  $\sup_{\gamma \in \overline{P_i}} |g_i| = \sup_{\gamma \in \overline{P_i}} g_i$ , we have

$$\begin{aligned}
\text{Var}(\Gamma_{\widetilde{F}^* \mu}) &\leq \sum_{i=1}^q \text{Var}_{\overline{T_i(P_i)}} \left[ (g_i \cdot \Gamma_{\mu_{\overline{F}}}) \circ T_i^{-1} \cdot \chi_{T(P_i)} \right] \\
&\leq \sum_{i=1}^q \text{Var}_{\overline{T_i(P_i)}} \left[ (g_i \cdot \Gamma_{\mu_{\overline{F}}}) \circ T_i^{-1} \right] \cdot \sup |\chi_{T(P_i)}| \\
&\quad + \sum_{i=1}^q \sup_{\overline{T_i(P_i)}} \| (g_i \cdot \Gamma_{\mu_{\overline{F}}}) \circ T_i^{-1} \|_W \cdot \text{var}(\chi_{T(P_i)}) \\
&\leq \sum_{i=1}^q \text{Var}_{\overline{P_i}}(g_i \cdot \Gamma_{\mu_{\overline{F}}}) + 2 \cdot \sup_{\overline{T_i(P_i)}} \| (g_i \cdot \Gamma_{\mu_{\overline{F}}}) \circ T_i^{-1} \|_W \\
&\leq \sum_{i=1}^q \text{var}_{\overline{P_i}}(g_i) \cdot \sup_{\overline{P_i}} \|\Gamma_{\mu_{\overline{F}}}\|_W + \text{Var}_{\overline{P_i}}(\Gamma_{\mu_{\overline{F}}}) \cdot \sup_{\overline{P_i}} |g_i| \\
&\quad + 2 \cdot \sum_{i=1}^q \sup_{\overline{P_i}} |g_i| \sup_{\overline{P_i}} \|\Gamma_{\mu_{\overline{F}}}\|_W \\
&\leq \sum_{i=1}^q \text{var}_{\overline{P_i}}(g_i) \cdot \sup_{\gamma \in \overline{P_i}} \|\Gamma_{\mu}(\gamma)\|_W + \text{Var}_{\overline{P_i}}(\Gamma_{\mu_{\overline{F}}}) \cdot \sup_{\overline{P_i}} |g_i| \\
&\quad + 2 \cdot \sum_{i=1}^q \sup_{\gamma \in \overline{P_i}} \|\Gamma_{\mu}(\gamma)\|_W \cdot \sup_{\overline{P_i}} |g_i| \\
&\leq \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \cdot \sup_{\gamma \in \overline{P_i}} \|\Gamma_{\mu}(\gamma)\|_W + \sup_{\overline{P_i}} g_i \cdot \text{Var}_{\overline{P_i}}(\Gamma_{\mu_{\overline{F}}}).
\end{aligned}$$

□

**9.11. Lemma.** *For all path  $\Gamma_{\mu}$ , where  $\mu \in \mathcal{BV}^+$ , it holds*

$$\text{Var}(\Gamma_{\widetilde{F}^* \mu}) \leq \beta \text{Var}(\Gamma_{\mu}) + K_3 |\phi_x|_{1,1}.$$

Where  $0 < \beta < 1$  comes from equation (60) of remark 9.4 and

$$K_3 = \max_{i=1, \dots, q} \left\{ \sup_{\overline{P_i}} g_i \right\} \text{var}^{\diamond}(G) + \max_{i=1, \dots, q} \left\{ \frac{\text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i}{m(P_i)} \right\}.$$

*Proof.* By lemma 9.9, remark 9.8, lemma 9.10, P1, equation (60) of remark 9.4 and by  $\sum_{i=1}^q \text{var}_{\overline{P_i}}^{\diamond} G = \text{var}^{\diamond}(G)$ , we get

$$\begin{aligned}
\text{Var}(\Gamma_{\tilde{F}^*\mu}) &\leq \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \sup_{\gamma \in \overline{P_i}} \|\mu|_\gamma\|_W + \sup_{\overline{P_i}} g_i \cdot \text{Var}_{\overline{P_i}}(\Gamma_{\mu_{\tilde{F}}}) \\
&\leq \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \left( \text{Var}_{\overline{P_i}}(\Gamma_\mu) + \frac{1}{m(\overline{P_i})} \int_{\overline{P_i}} \|\mu|_\gamma\|_W dm(\gamma) \right) \\
&\quad + \sum_{i=1}^q \sup_{\overline{P_i}} g_i \left( \text{Var}_{\overline{P_i}}(\Gamma_\mu) + \text{var}_{\overline{P_i}}^\diamond(G) |\phi_x|_\infty \right) \\
&\leq \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 3 \sup_{\overline{P_i}} g_i \right] \text{Var}_{\overline{P_i}}(\Gamma_\mu) \\
&\quad + \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \frac{1}{m(\overline{P_i})} \int_{\overline{P_i}} \|\mu|_\gamma\|_W dm(\gamma) \\
&\quad + |\phi_x|_\infty \max_{i=1, \dots, q} \left\{ \sup_{\overline{P_i}} g_i \right\} \text{var}^\diamond(G) \\
&\leq \sum_{i=1}^q \left[ \text{var}_{\overline{P_i}}(g_i) + 3 \sup_{\overline{P_i}} g_i \right] \text{Var}_{\overline{P_i}}(\Gamma_\mu) \\
&\quad + \max_{i=1, \dots, q} \left\{ \frac{\text{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i}{m(\overline{P_i})} \right\} |\phi_x|_1 \\
&\quad + |\phi_x|_\infty \max_{i=1, \dots, q} \left\{ \sup_{\overline{P_i}} g_i \right\} \text{var}^\diamond(G) \\
&\leq \beta \text{Var}(\Gamma_\mu) + K_3 |\phi_x|_\infty \\
&\leq \beta \text{Var}(\Gamma_\mu) + K_3 |\phi_x|_{1,1}.
\end{aligned}$$

□

Taking the infimum over all paths  $\Gamma_\mu^\omega \in \Gamma_\mu$  we arrive at the following

**9.12. Proposition.** *For every  $\mu \in \mathcal{BV}^+$ , it holds*

$$\text{Var}(\tilde{F}^*\mu) \leq \beta \text{Var}(\mu) + K_3 |\phi_x|_{1,1}.$$

We are ready to prove Theorem 9.2.

*Proof.* (of Theorem 9.2) Inequality (41) gives us

$$(65) \quad |P_T^n f|_{1,1} \leq B_3 \beta_2^n |f|_{1,1} + C_2 |f|_1, \quad \forall n, \quad \forall f \in BV_{1,1},$$

for  $B_3, C_2 > 0$  and  $0 < \beta_2 < 1$ . Then, since  $|f|_1 \leq |f|_{1,1}$ , it holds

$$(66) \quad |P_T^n f|_{1,1} \leq K_2 |f|_{1,1}, \quad \forall n, \quad \forall f \in BV_{1,1},$$

where

$$(67) \quad K_2 = B_3 + C_2.$$

In particular, inequality (66) holds if we replace  $f$  by  $\phi_x = \frac{d(\pi_x^* \mu)}{dm}$  for each  $\mu \in \mathcal{BV}^+$ .

By inequality (66), Theorem 9.12 and a straightforward induction we have

$$(68) \quad \text{Var}(\tilde{F}^{*n} \mu) \leq \beta^n \text{Var}(\mu) + K_3 \max\{K_2, 1\} \sum_{i=0}^{n-1} \beta^i |\phi_x|_{1,1}, \quad \forall n \geq 0.$$

We finish the proof by setting  $\lambda_0 := \beta$  and

$$(69) \quad C_0 := \max \left\{ 1, \frac{K_3 \max\{K_2, 1\}}{1 - \beta} \right\}.$$

□

By remark 9.3 we immediately get

**9.13. Proposition.** *Let  $F_L(x, y) = (T_L(x), G_L(x, y))$  be a BV Lorenz-like map and consider  $\mu \in \mathcal{BV}^+$ . Then, there are  $C_0, 0 < \lambda_0 < 1$  and  $k \in \mathbb{N}$  such that for all  $\mu \in \mathcal{BV}^+$  and all  $n \geq 1$  it holds (denote  $\tilde{F} := F^k$ )*

$$(70) \quad \text{Var}(\tilde{F}^{*n} \mu) \leq C_0 \lambda_0^n \text{Var}(\mu) + C_0 |\phi_x|_{1,1}.$$

**9.14. Proposition.** *Let  $\{F_\delta\}_\delta, F_\delta = (T_\delta, G_\delta)$  be a Uniform BV Lorenz-like family (definition (8.7)) and let  $f_\delta$  be the unique  $F_\delta$ -invariant probability in  $\mathcal{BV}_{1,1}^\infty$ . Then, there exists  $\overline{C}_0 > 0$  such that*

$$(71) \quad \text{Var}(f_\delta) \leq \overline{C}_0,$$

for all  $\delta \in [0, 1)$ .

*Proof.*

**9.15. Lemma.** *Let  $\{T_\delta\}_{\delta \in [0,1]}$  be a family of piecewise expanding maps satisfying definition 8.5, item (1), item (2) and item (3) of UBV4 (see definition 8.7). Then, there is  $k$  (which does not depends on  $\delta$ ) such that*

$$(72) \quad \text{var} g_{i,\delta}^{(k)} + 2 \sup g_{i,\delta}^{(k)} \leq \beta < 1, \quad \forall P_i \in \mathcal{P}.$$

*Proof.* (of the Lemma)

Set  $k_0 = \inf_{\delta \in [0,1]} n_0(\delta)$ ,  $\beta_0 = \inf_{\delta \in [0,1]} \{\lambda_1(\delta)\}$  (where  $\lambda_1(\delta)$  comes from item (1) of UBV4) and suppose that  $n > n_0(\delta)$  (see item (3) of UBV4) for all  $\delta \in [0, 1)$ . Also set  $n = qn_0 + r_0$  (remember  $n_0$  depends on  $\delta$ ),  $\tilde{T}_\delta := T_\delta^{n_0}$  and  $y = T_\delta^{r_0}(x)$ .

For all  $n \geq 1$ , let  $\mathcal{P}^{(n)}$  be the partition of  $I$  s.t.  $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$  if and only if  $\mathcal{P}^{(1)}(T_\delta^j(x)) = \mathcal{P}^{(1)}(T_\delta^j(y))$  for all  $j = 0, \dots, n$ , where  $\mathcal{P}^{(1)} = \mathcal{P}$ .

Given  $x \in P_i \in \mathcal{P}^{(n)} = \{P_1, \dots, P_q\}$  we have

$$\begin{aligned} |T_\delta^{n'}(x)| &= |(\tilde{T}_\delta^q)'(y)| \\ &\geq \beta_0^q. \end{aligned}$$

Then,

$$\begin{aligned}
g_{\delta,i}^{(n)}(x) &\leq \left(\frac{1}{\beta_0}\right)^q \\
&\leq (\beta_0) \cdot \left(\left(\frac{1}{\beta_0}\right)^{\frac{1}{k_0}}\right)^n.
\end{aligned}$$

Then, there are  $C_5 > 0$  and  $0 < \lambda_4 < 1$  (both do not depend on  $\delta$  or  $i$ ) such that

$$(73) \quad \sup g_{\delta,i}^{(n)} \leq C_5 \lambda_4^n,$$

where  $C_5 := \beta_0$  and  $\lambda_4 := \left(\frac{1}{\beta_0}\right)^{\frac{1}{k_0}}$ .

Now, set  $C_6 := \max\{C_4, C_5\}$  (see UB4). By UB4 we have  $\text{var } g_{\delta,i} \leq C_6$  for all  $i$  and all  $\delta \in [0, 1]$ . Thus, for all  $n \geq 1$  it holds (see [27], page 41, equation (3.1))

$$(74) \quad \text{var}(g_{\delta,i}^{(n)}) \leq C_7 \lambda_5^n \quad \forall \delta \in [0, 1] \text{ and } \forall i = 1, \dots, q,$$

where  $\lambda_5 \in (\lambda_4, 1)$  and  $C_7 := \sup_{n \geq 1} \left\{ \frac{C_6^3}{\lambda_4} n \left(\frac{\lambda_4}{\lambda_5}\right)^n \right\}$ . And the lemma is proved.  $\square$

Now, applying Proposition 8.16 to each  $F_\delta$  we get  $\text{Var}(f_\delta) \leq 2C_{0,\delta}$ , where (see equation (69) and equation (67))

$$(75) \quad C_{0,\delta} = \max \left\{ 1, \frac{K_{3,\delta} \max\{K_{2,\delta}, 1\}}{1 - \lambda_1(\delta)} \right\},$$

$$(76) \quad K_{3,\delta} = \max_i \left\{ \sup_{\overline{P}_i} g_{i,\delta} \right\} \text{var}^\diamond(G_\delta) + \max_i \left\{ \frac{\text{var}_{\overline{P}_i}(g_{i,\delta}) + 2 \sup_{\overline{P}_i} g_{i,\delta}}{m(P_i)} \right\}$$

and  $K_{2,\delta} = 2D$  by (UBV1) of definition 8.7. By Lemma 9.15 we get  $\sup_\delta \frac{K_{3,\delta}}{1 - \lambda_1(\delta)} < \infty$ . Therefore, we finish the proof of Proposition 9.14 by setting  $\overline{C}_0 = 2 \sup_\delta C_{0,\delta}$ .  $\square$

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(Rafael Lucena) UNIVERSIDADE FEDERAL DO RIO DE JANEIRO - UFRJ AND UNIVERSITÀ DI PISA - UNIFI

E-mail address: [rafael.lucena@im.ufal.br](mailto:rafael.lucena@im.ufal.br)

URL: [www.mathlucena.blogspot.com](http://www.mathlucena.blogspot.com)

(Stefano Galatolo) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA - UNIFI

E-mail address: [stefano.galatolo@unipi.it](mailto:stefano.galatolo@unipi.it)

URL: <http://users.dma.unipi.it/galatolo/>